More Natural Derivations for Priest, *An Introduction to Non-Classical Logic, 2nd ed.*

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September 7, 2008

In [7] I produced natural derivation systems, including demonstration of soundness and completeness, for each of the logics described in the first edition of Priest, *An Introduction to Non-Classical Logic* [3]. The first edition of Priest's book is Part I of the second edition. Eventually, I hope to complete the project, providing natural derivation systems for the quantified versions in Part II. In the meantime, without including parts for soundness and completeness, this document simply extends the previous paper to account for additions and changes in the first part of the new edition.

Thus, as before, I provide an alternative or supplement to the semantic tableaux of his text. Some of the derivation systems may also be of interest in their own right. They are all Fitch-style systems on the model of [1, 6], and many other places. Though a classical system is presented for chapter 1, prior acquaintance with some such system is assumed. Associated goal-directed derivation strategies are discussed extensively in [6, chapter 6].

Except that some chapters are collapsed, there are sections for each chapter in the first part of Priest's book, with an additional section on four-valued relevant logic. In each case, (i) the language is briefly described and key semantic definitions stated, and (ii) the derivation system is presented with a few examples given. For those with interest, demonstration of soundness and completeness should be straightforward given background and strategy from the published paper.

*Thanks to all!

Contents

1 Classical Logic: *CL* (ch. 1)

1.1 Language / Semantic Notions

LCL The LANGUAGE consists of propositional parameters $p_0, p_1 \ldots$ combined in the usual way with the operators, $\neg$, $\wedge$, $\vee$, $\supset$, and $\equiv$. So each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$ and $(A \equiv B)$.

ICL An INTERPRETATION is a function v which assigns to each propositional parameter either 1 (true) or 0 (false).

TCL For complex expressions,

($\neg$) $v(\neg A) = 1$ if $v(A) = 0$, and 0 otherwise.

($\wedge$) $v(A \wedge B) = 1$ if $v(A) = 1$ and $v(B) = 1$, and 0 otherwise.

($\vee$) $v(A \vee B) = 1$ if $v(A) = 1$ or $v(B) = 1$, and 0 otherwise.

($\supset$) $v(A \supset B) = 1$ if $v(A) = 0$ or $v(B) = 1$, and 0 otherwise.

($\equiv$) $v(A \equiv B) = 1$ if $v(A) = v(B)$, and 0 otherwise.

For a set Γ of formulas, $v(\Gamma) = 1$ iff $v(A) = 1$ for each $A \in \Gamma$; then,

VCL $\Gamma \models_{CL} A$ iff there is no *CL* interpretation v such that $v(\Gamma) = 1$ and $v(A) = 0$.

1.2 Natural Derivations: *NCL*

NCL is just the sentential portion of the system *ND* from [6, chapter 6]. Refer to that source for examples and further discussion (compare, e.g., [1]). Every line of a derivation is a premise, an assumption, or justified from previous lines by a rule. The rules include *introduction* and *exploitation* rules for each operator, and *reiteration*. In the parenthetical "exit strategy" for assumptions, 'c' indicates a contradiction is to be sought, 'g' a goal at the bottom of the scope line.

R (*reiteration*)

a	P	
	P	a R

$\neg$**I** (*negation intro*)

a	$\mid$ P	A (c, $\neg$I)
	$\mid$ Q	
b	$\mid$ $\neg Q$	
	$\neg P$	a-b $\neg$I

$\neg$**E** (*negation exploit*)

a	$\mid$ $\neg P$	A (c, $\neg$E)
	$\mid$ Q	
b	$\mid$ $\neg Q$	
	P	a-b $\neg$E

$\wedge$**I** (*conjunction intro*)

a	P	
b	Q	
	$P \wedge Q$	a,b $\wedge$I

$\wedge$**E** (*conjunction exploit*)

a	$P \wedge Q$	
	P	a $\wedge$E

$\wedge$**E** (*conjunction exploit*)

a	$P \wedge Q$	
	Q	a $\wedge$E

$\vee$**I** (*disjunction intro*)

a	P	
	$P \vee Q$	a $\vee$I

$\vee$**I** (*disjunction intro*)

a	P	
	$Q \vee P$	a $\vee$I

$\vee$**E** (*disjunction exploit*)

a	$P \vee Q$	
b	$\mid$ P	A (g, a $\vee$E)
c	$\mid$ R	
d	$\mid$ Q	A (g, a $\vee$E)
e	$\mid$ R	
	R	a,b-c,d-e $\vee$E

$\supset$**I** (*conditional intro*)

a	$\mid$ P	A (g, $\supset$I)
b	$\mid$ Q	
	$P \supset Q$	a-b $\supset$I

$\supset$**E** (*conditional exploit*)

a	$P \supset Q$	
b	P	
	Q	a,b $\supset$E

≡I (*biconditional intro*)

a	$\quad P$	A (g, ≡I)
b	$\quad Q$	
c	$\quad Q$	A (g, ≡I)
d	$\quad P$	
	$P \equiv Q$	a-b, c-d ≡I

≡E (*biconditional exploit*)

a	$P \equiv Q$	
b	P	
	Q	a,b ≡E

≡E (*biconditional exploit*)

a	$P \equiv Q$	
b	Q	
	P	a,b ≡E

NCL $\Gamma \vdash_{NCL} A$ iff there is an *NCL* derivation of A from the members of Γ.

As derived rules, we accept the following "ordinary" and "two-way" rules. The "two-way" rules are usually presented as *replacement* rules. Insofar as we will not have much call to use then that way, in order to streamline demonstrations of soundness, we treat them just as ordinary rules which work in either direction – where it is trivial that the rules are in fact derived in this sense from the rules of *NCL*.

Ordinary Derived Rules

modus tollens

MT

$P \supset Q$
$\neg Q$
$\neg P$

negated biconditional

NB

$P \equiv Q$	$P \equiv Q$
$\neg P$	$\neg Q$
$\neg Q$	$\neg P$

disjunctive syllogism

DS

$P \vee Q$	$P \vee Q$
$\neg P$	$\neg Q$
Q	P

Two-way Derived Rules

DN	$P \;\triangleleft\triangleright\; \neg\neg P$	*double negation*
Com	$P \wedge Q \;\triangleleft\triangleright\; Q \wedge P$ $P \vee Q \;\triangleleft\triangleright\; Q \vee P$	*commutation*
Assoc	$P \wedge (Q \wedge R) \;\triangleleft\triangleright\; (P \wedge Q) \wedge R$ $P \vee (Q \vee R) \;\triangleleft\triangleright\; (P \vee Q) \vee R$	*association*
Idem	$P \;\triangleleft\triangleright\; P \wedge P$ $P \;\triangleleft\triangleright\; P \vee P$	*idempotence*
Impl	$P \supset Q \;\triangleleft\triangleright\; \neg P \vee Q$ $\neg P \supset Q \;\triangleleft\triangleright\; P \vee Q$	*implication*
Trans	$P \supset Q \;\triangleleft\triangleright\; \neg Q \supset \neg P$	*transposition*

DeM	$\neg(P \wedge Q) \;\triangleleft\triangleright\; \neg P \vee \neg Q$	*De Morgan*
	$\neg(P \vee Q) \;\triangleleft\triangleright\; \neg P \wedge \neg Q$	
Exp	$P \supset (Q \supset R) \;\triangleleft\triangleright\; (P \wedge Q) \supset R$	*exportation*
Equiv	$P \equiv Q \;\triangleleft\triangleright\; (P \supset Q) \wedge (Q \supset P)$	*equivalence*
	$P \equiv Q \;\triangleleft\triangleright\; (P \wedge Q) \vee (\neg P \wedge \neg Q)$	
Dist	$P \wedge (Q \vee R) \;\triangleleft\triangleright\; (P \wedge Q) \vee (P \wedge R)$	*distribution*
	$P \vee (Q \wedge R) \;\triangleleft\triangleright\; (P \vee Q) \wedge (P \vee R)$	

Examples. Here are derivations to demonstrate the first form of Impl (among the relatively difficult of derivations for the derived rules).

$\neg P \vee Q \vdash_{NCL} P \supset Q$

1	$\mid\ \neg P \vee Q$	P
2	$\mid\ \mid\ \neg P$	A (g, 1 $\vee$E)
3	$\mid\ \mid\ \mid\ P$	A (g, $\supset$I)
4	$\mid\ \mid\ \mid\ \mid\ \neg Q$	A (c, $\neg$E)
5	$\mid\ \mid\ \mid\ \mid\ \neg P$	2 R
6	$\mid\ \mid\ \mid\ \mid\ P$	3 R
7	$\mid\ \mid\ \mid\ Q$	4-6 $\neg$E
8	$\mid\ \mid\ P \supset Q$	3-7 $\supset$I
9	$\mid\ \mid\ Q$	A (g, 1 $\vee$E)
10	$\mid\ \mid\ \mid\ P$	A (g, $\supset$I)
11	$\mid\ \mid\ \mid\ Q$	9 R
12	$\mid\ \mid\ P \supset Q$	10-11 $\supset$I
13	$\mid\ P \supset Q$	1,2-8,9-12 $\vee$E

$P \supset Q \vdash_{NCL} \neg P \vee Q$

1	$\mid\ P \supset Q$	P
2	$\mid\ \mid\ \neg(\neg P \vee Q)$	A (c, $\neg$E)
3	$\mid\ \mid\ \mid\ P$	A (c, $\neg$I)
4	$\mid\ \mid\ \mid\ Q$	1,3 $\supset$E
5	$\mid\ \mid\ \mid\ \neg P \vee Q$	4 $\vee$I
6	$\mid\ \mid\ \mid\ \neg(\neg P \vee Q)$	2 R
7	$\mid\ \mid\ \neg P$	3-6 $\neg$I
8	$\mid\ \mid\ \neg P \vee Q$	7 $\vee$I
9	$\mid\ \mid\ \neg(\neg P \vee Q)$	2 R
10	$\mid\ \neg P \vee Q$	2-9 $\neg$E

2 Normal Modal Logics: $\boldsymbol{K}\alpha$, $\boldsymbol{K}^t_\alpha$ (ch. 2,3)

2.1 Language / Semantic Notions

LKα For the $K\alpha$ systems, the VOCABULARY consists of propositional parameters $p_0, p_1 \ldots$ with the operators, $\neg$, $\wedge$, $\vee$, $\supset$, $\equiv$, $\Box$ and $\Diamond$. Each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$, $(A \equiv B)$, $\Box A$ and $\Diamond A$.

IKα For any of these systems except Kυ, an INTERPRETATION is a triple $\langle W, R, v\rangle$ where W is a set of worlds, R is a subset of $W^2 = W \times W$, and v is a function such that for any $w \in W$ and p, $v_w(p) = 1$ or $v_w(p) = 0$. For $x, y, z \in W$, where α is empty or indicates some combination of the following constraints,

η	For any x, there is a y such that xRy	extendability
ρ	for all x, xRx	reflexivity
σ	for all x, y, if xRy then yRx	symmetry
τ	for all x, y, z, if xRy and yRz then xRz	transitivity

$\langle W, R, v\rangle$ is a $K\alpha$ interpretation when R meets the constraints from α.

TK For complex expressions,

($\neg$) $v_w(\neg A) = 1$ if $v_w(A) = 0$, and 0 otherwise.

($\wedge$) $v_w(A \wedge B) = 1$ if $v_w(A) = 1$ and $v_w(B) = 1$, and 0 otherwise.

($\vee$) $v_w(A \vee B) = 1$ if $v_w(A) = 1$ or $v_w(B) = 1$, and 0 otherwise.

($\supset$) $v_w(A \supset B) = 1$ if $v_w(A) = 0$ or $v_w(B) = 1$, and 0 otherwise.

($\equiv$) $v_w(A \equiv B) = 1$ if $v_w(A) = v_w(B)$, and 0 otherwise.

($\Diamond$) $v_w(\Diamond A) = 1$ if some $x \in W$ such that wRx has $v_x(A) = 1$, and 0 otherwise.

($\Box$) $v_w(\Box A) = 1$ if all $x \in W$ such that wRx have $v_x(A) = 1$, and 0 otherwise.

For a set Γ of formulas, $v_w(\Gamma) = 1$ iff $v_w(A) = 1$ for each $A \in \Gamma$; then,

VKα $\Gamma \models_{K\alpha} A$ iff there is no $K\alpha$ interpretation $\langle W, R, v\rangle$ and $w \in W$ such that $v_w(\Gamma) = 1$ and $v_w(A) = 0$.

System Kυ. For Kυ either accept the constraint, (υ) for all x, y, xRy. Then let everything work as before. Otherwise simplify the semantics: An interpretation is just $\langle W, v\rangle$. For TK($\Box$) and TK($\Diamond$) substitute,

TK ($\Diamond$)$_\upsilon$ $v_w(\Diamond A) = 1$ iff for some $x \in W$, $v_x(A) = 1$.

($\Box$)$_\upsilon$ $v_w(\Box A) = 1$ iff for all $x \in W$, $v_x(A) = 1$.

then,

VKυ $\Gamma \models_{K\upsilon} A$ iff there is no $K\upsilon$ interpretation $\langle W, v\rangle$ and $w \in W$ such that $v_w(\Gamma) = 1$ and $v_w(A) = 0$.

System $\mathbf{K}^t_\alpha$. Extend the language so that in place of $\Box$ and $\Diamond$ there are operators, [F], ⟨F⟩, [P], and ⟨P⟩. If A is a formuula, so are [F]A, ⟨F⟩A, [P]A, and ⟨P⟩A. Interpretations, with constraints ρ, σ, τ are as in Kα. Allow also,

η	For any x, there is a y such that xRy	extendability
η'	For any x, there is a y such that yRx	backward extendibility
δ	If xRy then for some z, xRz and zRy	denseness
φ	If xRy and xRz then yRz or $y = z$ or zRy	forward convergence
β	If yRx and zRx then yRz or $y = z$ or zRy	backward convergence

Then TK^t includes,

([F]) $v_w([\mathrm{F}]A) = 1$ iff all $x \in W$ such that wRx have $v_x(A) = 1$.

([P]) $v_w([\mathrm{P}]A) = 1$ iff all $x \in W$ such that xRw have $v_x(A) = 1$.

(⟨F⟩) $v_w(\langle\mathrm{F}\rangle A) = 1$ iff some $x \in W$ such that wRx has $v_x(A) = 1$.

(⟨P⟩) $v_w(\langle\mathrm{P}\rangle A) = 1$ iff some $x \in W$ such that xRw has $v_x(A) = 1$.

and VK$^t_\alpha$ works in the usual way.

2.2 Natural Derivations: *NK*α

Where s is any integer, let A_s be a SUBSCRIPTED FORMULA. For subscripts s and t allow also expressions of the sort, $s.t$. As in Priest, intuitively, subscripts indicate worlds, where A_s is true or false at world s, and $s.t$ just in case world s has access to world t. Derivation rules apply to these expressions. Rules for $\neg$, $\wedge$, $\vee$, $\supset$, and $\equiv$ are like ones from before, but with consistent subscripts. Rules for $\Box$ and $\Diamond$ are new.[1]

R
$$\begin{array}{|l} P_s \\ \\ P_s \end{array}$$

¬I
$$\begin{array}{|l} \begin{array}{|l} P_s \\ \hline \\ Q_t \\ \neg Q_t \end{array} \\ \neg P_s \end{array}$$

¬E
$$\begin{array}{|l} \begin{array}{|l} \neg P_s \\ \hline \\ Q_t \\ \neg Q_t \end{array} \\ P_s \end{array}$$

∧I
$$\begin{array}{|l} P_s \\ Q_s \\ \\ (P \wedge Q)_s \end{array}$$

∧E
$$\begin{array}{|l} (P \wedge Q)_s \\ \\ P_s \end{array}$$

∧E
$$\begin{array}{|l} (P \wedge Q)_s \\ \\ Q_s \end{array}$$

[1]There is no uniformity about how to do natural deduction in modal logic. Most avoid subscripts altogether. Another option uses subscripts of the sort $i.j \ldots k$ (cf. prefixes on tableaux in [2]); the result is elegant, but not so flexible as this account inspired by Priest, and we will need the flexibility, as we approach increasingly complex systems.

$$\vee\mathbf{I}\ \begin{array}{|l} P_s \\ \\ (P \vee Q)_s \end{array} \qquad \vee\mathbf{I}\ \begin{array}{|l} P_s \\ \\ (Q \vee P)_s \end{array} \qquad \vee\mathbf{E}\ \begin{array}{|l} (P \vee Q)_s \\ \begin{array}{|l} \underline{P_s} \\ \\ R_t \end{array} \\ \begin{array}{|l} \underline{Q_s} \\ \\ R_t \end{array} \\ R_t \end{array}$$

$$\supset\mathbf{I}\ \begin{array}{|l} \begin{array}{|l} \underline{P_s} \\ \\ Q_s \end{array} \\ (P \supset Q)_s \end{array} \qquad \supset\mathbf{E}\ \begin{array}{|l} (P \supset Q)_s \\ P_s \\ \\ Q_s \end{array}$$

$$\equiv\mathbf{I}\ \begin{array}{|l} \begin{array}{|l} \underline{P_s} \\ \\ Q_s \end{array} \\ \begin{array}{|l} \underline{Q_s} \\ \\ P_s \end{array} \\ (P \equiv Q)_s \end{array} \qquad \equiv\mathbf{E}\ \begin{array}{|l} (P \equiv Q)_s \\ P_s \\ \\ Q_s \end{array} \qquad \equiv\mathbf{E}\ \begin{array}{|l} (P \equiv Q)_s \\ Q_s \\ \\ P_s \end{array}$$

$$\Box\mathbf{I}\ \begin{array}{|l} \begin{array}{|l} \underline{s.t} \\ \\ P_t \end{array} \\ \Box P_s \end{array} \qquad \Box\mathbf{E}\ \begin{array}{|l} \Box P_s \\ s.t \\ \\ P_t \end{array} \qquad \Diamond\mathbf{I}\ \begin{array}{|l} P_t \\ s.t \\ \\ \Diamond P_s \end{array} \qquad \Diamond\mathbf{E}\ \begin{array}{|l} \Diamond P_s \\ \begin{array}{|l} s.t \\ \underline{P_t} \\ \\ Q_u \end{array} \\ Q_u \end{array}$$

$\Box$I: where t does not appear in any undischarged premise or assumption

$\Diamond$E: where t does not appear in any undischarged premise or assumption and is not u

These are the rules of *NK*. Other systems $NK\alpha$ add from the following, for *access manipulation*, according to constraints in α.

$$\mathbf{AM}\eta\ \begin{array}{|l} \begin{array}{|l} \underline{s.t} \\ \\ P_u \end{array} \\ P_u \end{array} \qquad \mathbf{AM}\rho\ \begin{array}{|l} \\ \\ s.s \end{array} \qquad \mathbf{AM}\sigma\ \begin{array}{|l} s.t \\ \\ t.s \end{array} \qquad \mathbf{AM}\tau\ \begin{array}{|l} s.t \\ t.u \\ \\ s.u \end{array}$$

AMη: where t does not appear in any undischarged premise or assumption and is not u

AMρ has no premise. In these systems, every subscript is 0, appears in a premise, or appears in the t-place of an accessible assumption for $\Box$I, $\Diamond$E, or AMη. Where Γ is a set of unsubscripted formulas, let Γ_0 be those same formulas, each with subscript 0. Then, NKα $\Gamma \vdash_{NK\alpha} A$ iff there is an $NK\alpha$ derivation of A_0 from the members of Γ_0.

Derived rules carry over from *NCL* as one would expect, with subscripts constant throughout. Thus, e.g.,

MT
$$\begin{array}{|l} (P \supset Q)_s \\ \neg Q_s \\ \\ \neg P_s \end{array}$$

Impl
$$(P \supset Q)_s \ \lhd\rhd \ (\neg P \vee Q)_s$$
$$(\neg P \supset Q)_s \ \lhd\rhd \ (P \vee Q)_s$$

Allow also the additional rule for *modal negation*,

MN
$$\Box P_s \ \lhd\rhd \ \neg\Diamond\neg P_s \qquad \neg\Box P_s \ \lhd\rhd \ \Diamond\neg P_s$$
$$\Diamond P_s \ \lhd\rhd \ \neg\Box\neg P_s \qquad \neg\Diamond P_s \ \lhd\rhd \ \Box\neg P_s$$

System NKυ. For $NK\upsilon$, eliminate expressions of the sort $s.t$ and rules for access manipulation. Let $\top$ be an arbitrary tautology (say, $p \supset p$). Then for $\Box$I, $\Box$E, $\Diamond$I and $\Diamond$E, substitute,

$\Box$**I**υ
$$\begin{array}{|l} \begin{array}{|l} \top_t \\ \hline \\ P_t \end{array} \\ \Box P_s \end{array}$$
where t does not appear in any undischarged premise or assumption

$\Box$**E**υ
$$\begin{array}{|l} \Box P_s \\ \\ P_t \end{array}$$

$\Diamond$**I**υ
$$\begin{array}{|l} P_t \\ \\ \Diamond P_s \end{array}$$

$\Diamond$**E**υ
$$\begin{array}{|l} \Diamond P_s \\ \begin{array}{|l} P_t \\ \hline \\ Q_u \end{array} \\ Q_u \end{array}$$
where t does not appear in any undischarged premise or assumption and is not u

System NK$^t_\alpha$. Rules for NK^t_α are like $NK\alpha$ except that there are the following rules for the new operators,

[F]**I**
$$\begin{array}{|l} \begin{array}{|l} s.t \\ \hline \\ A_t \end{array} \\ [\mathrm{F}]A_s \end{array}$$
where t does not appear in any undischarged premise or assumption

[F]**E**
$$\begin{array}{|l} [\mathrm{F}]A_s \\ s.t \\ \\ A_t \end{array}$$

$\langle$F$\rangle$**I**
$$\begin{array}{|l} A_t \\ s.t \\ \\ \langle \mathrm{F}\rangle A_s \end{array}$$

$\langle$F$\rangle$**E**
$$\begin{array}{|l} \langle \mathrm{F}\rangle A_s \\ \begin{array}{|l} s.t \\ A_t \\ \hline \\ Q_u \end{array} \\ Q_u \end{array}$$
where t does not appear in any undischarged premise or assumption and is not u

[P]I

$$\begin{array}{|l} \begin{array}{|l} s.t \\ \hline \\ A_s \end{array} \\ [\mathrm{P}]A_t \end{array}$$

where s does not appear in any undischarged premise or assumption

[P]E

$$\begin{array}{|l} [\mathrm{P}]A_t \\ s.t \\ \\ A_s \end{array}$$

⟨P⟩I

$$\begin{array}{|l} A_s \\ s.t \\ \\ \langle \mathrm{P}\rangle A_t \end{array}$$

⟨P⟩E

$$\begin{array}{|l} \langle \mathrm{P}\rangle A_t \\ \begin{array}{|l} s.t \\ A_s \\ \hline \\ Q_u \end{array} \\ Q_u \end{array}$$

where s does not appear in any undischarged premise or assumption and is not u

And the following rules corresponding to the new constraints on access. Where $\mathcal{A}(i)$ is any expression in which i appears, and $\mathcal{A}(j)$ is the same expression with j substituted for i,

AMη

$$\begin{array}{|l} \begin{array}{|l} s.t \\ \hline \\ P_u \end{array} \\ P_u \end{array}$$

where t does not appear in any undischarged premise or assumption and is not u

AMη'

$$\begin{array}{|l} \begin{array}{|l} s.t \\ \hline \\ P_u \end{array} \\ P_u \end{array}$$

where s does not appear in any undischarged premise or assumption and is not u

=E

$$\begin{array}{|l} s = t \\ \mathcal{A}(s) \\ \\ \mathcal{A}(t) \end{array} \qquad \begin{array}{|l} t = s \\ \mathcal{A}(s) \\ \\ \mathcal{A}(t) \end{array}$$

AMδ

$$\begin{array}{|l} s.t \\ \begin{array}{|l} s.a \\ a.t \\ \hline \\ Q_u \end{array} \\ Q_u \end{array}$$

where a does not appear in any undischarged premise or assumption

AMφ

$$\begin{array}{|l} r.s \\ r.t \\ \begin{array}{|l} s.t \\ \hline \\ Q_u \end{array} \\ \begin{array}{|l} s = t \\ \hline \\ Q_u \end{array} \\ \begin{array}{|l} t.s \\ \hline \\ Q_u \end{array} \\ Q_u \end{array}$$

AMβ

$$\begin{array}{|l} s.r \\ t.r \\ \begin{array}{|l} s.t \\ \hline \\ Q_u \end{array} \\ \begin{array}{|l} s = t \\ \hline \\ Q_u \end{array} \\ \begin{array}{|l} t.s \\ \hline \\ Q_u \end{array} \\ Q_u \end{array}$$

Examples. Here are derivations to exhibit left-hand forms of the rule for modal negation as derived in NK (and so any $NK\alpha$).

$\neg\Diamond\neg P \vdash_{NK} \Box P$

1	$\mid \neg\Diamond\neg P_0$	P
2	$\mid\ \mid 0.1$	A (g, $\Box$I)
3	$\mid\ \mid\ \mid \neg P_1$	A (c, $\neg$E)
4	$\mid\ \mid\ \mid \Diamond\neg P_0$	2,3 $\Diamond$I
5	$\mid\ \mid\ \mid \neg\Diamond\neg P_0$	1 R
6	$\mid\ \mid P_1$	3-5 $\neg$E
7	$\mid \Box P_0$	2-6 $\Box$I

$\Box P \vdash_{NK} \neg\Diamond\neg P$

1	$\mid \Box P_0$	P
2	$\mid\ \mid \Diamond\neg P_0$	A (c, $\neg$I)
3	$\mid\ \mid\ \mid 0.1$	A (g, 2 $\Diamond$E)
4	$\mid\ \mid\ \mid \neg P_1$	
5	$\mid\ \mid\ \mid\ \mid \Diamond\neg P_0$	A (c, $\neg$I)
6	$\mid\ \mid\ \mid\ \mid \neg P_1$	4 R
7	$\mid\ \mid\ \mid\ \mid P_1$	1,3 $\Box$E
8	$\mid\ \mid\ \mid \neg\Diamond\neg P_0$	5-7 $\neg$I
9	$\mid\ \mid \neg\Diamond\neg P_0$	2,3-8 $\Diamond$E
10	$\mid\ \mid \Diamond\neg P_0$	2 R
11	$\mid \neg\Diamond\neg P_0$	2-10 $\neg$I

$\neg\Box\neg P \vdash_{NK} \Diamond P$

1	$\mid \neg\Box\neg P_0$	P
2	$\mid\ \mid \neg\Diamond P_0$	A (c, $\neg$E)
3	$\mid\ \mid\ \mid 0.1$	A (g, $\Box$I)
4	$\mid\ \mid\ \mid\ \mid P_1$	A (c, $\neg$I)
5	$\mid\ \mid\ \mid\ \mid \Diamond P_0$	3,4 $\Diamond$I
6	$\mid\ \mid\ \mid\ \mid \neg\Diamond P_0$	2 R
7	$\mid\ \mid\ \mid \neg P_1$	4-6 $\neg$I
8	$\mid\ \mid \Box\neg P_0$	3-7 $\Box$I
9	$\mid\ \mid \neg\Box\neg P_0$	1 R
10	$\mid \Diamond P_0$	2-9 $\neg$E

$\Diamond P \vdash_{NK} \neg\Box\neg P$

1	$\mid \Diamond P_0$	P
2	$\mid\ \mid 0.1$	A (g, 1 $\Diamond$E)
3	$\mid\ \mid P_1$	
4	$\mid\ \mid\ \mid \Box\neg P_0$	A (c, $\neg$I)
5	$\mid\ \mid\ \mid \neg P_1$	2,4 $\Box$E
6	$\mid\ \mid\ \mid P_1$	3 R
7	$\mid\ \mid \neg\Box\neg P_0$	4-6 $\neg$I
8	$\mid \neg\Box\neg P_0$	1,2-7 $\Diamond$E

And some derivations in the other other systems,

$\vdash_{NK\eta} \Box P \supset \Diamond P$

1	$\mid\ \mid \Box P_0$	A (g, $\supset$I)
2	$\mid\ \mid\ \mid 0.1$	A (g, AMη)
3	$\mid\ \mid\ \mid P_1$	1,2 $\Box$E
4	$\mid\ \mid\ \mid \Diamond P_0$	2,3 $\Diamond$I
5	$\mid\ \mid \Diamond P_0$	2-4 AMη
6	$\mid (\Box P \supset \Diamond P)_0$	1-5 $\supset$I

$\vdash_{NK\rho} \Box P \supset P$

1	$\mid\ \mid \Box P_0$	A (g, $\supset$I)
2	$\mid\ \mid 0.0$	AMρ
3	$\mid\ \mid P_0$	1,2 $\Box$E
4	$\mid (\Box P \supset P)_0$	1-3 $\supset$I

$\vdash_{NK\sigma} P \supset \Box\Diamond P$

1	P_0	A (g, $\supset$I)
2	0.1	A (g, $\Box$I)
3	1.0	2 AMσ
4	$\Diamond P_1$	1,3 $\Diamond$I
5	$\Box\Diamond P_0$	2-4 $\Box$I
6	$(P \supset \Box\Diamond P)_0$	1-5 $\supset$I

$\vdash_{NK\tau} \Box P \supset \Box\Box P$

1	$\Box P_0$	A (g, $\supset$I)
2	0.1	A (g, $\Box$I)
3	1.2	A (g, $\Box$I)
4	0.2	2,3 AMτ
5	P_2	1,4 $\Box$E
6	$\Box P_1$	3-5 $\Box$I
7	$\Box\Box P_0$	2-6 $\Box$I
8	$(\Box P \supset \Box\Box P)_0$	1-7 $\supset$I

$\vdash_{NK\sigma\tau} \Diamond P \supset \Box\Diamond P$

1	$\Diamond P_0$	A (g, $\supset$I)
2	0.1	A (g, 1 $\Diamond$E)
3	P_1	
4	0.2	A (g, $\Box$I)
5	2.0	4 AMσ
6	2.1	5,2 AMτ
7	$\Diamond P_2$	3,6 $\Diamond$I
8	$\Box\Diamond P_0$	4-7 $\Box$I
9	$\Box\Diamond P_0$	1,2-8 $\Diamond$E
10	$(\Diamond P \supset \Box\Diamond P)_0$	1-9 $\supset$I

$\vdash_{NK\upsilon} \Diamond P \supset \Box\Diamond P$

1	$\Diamond P_0$	A (g, $\supset$I)
2	P_1	A (g, 1 $\Diamond$E)
3	$\top_2$	A (g, $\Box$I)
4	$\Diamond P_2$	2 $\Diamond$I
5	$\Box\Diamond P_0$	3-4 $\Box$I
6	$\Box\Diamond P_0$	1,2-5 $\Diamond$E
7	$(\Diamond P \supset \Box\Diamond P)_0$	1-6 $\supset$I

$[P][P]A \vdash_{NK^t_\delta} [P]A$

1	$[P][P]A_0$	P
2	$\quad 1.0$	A (g, [P]I)
3	$\quad\quad 1.2$	A (g, AMδ)
4	$\quad\quad 2.0$	
5	$\quad\quad [P]A_2$	1,4 [P]E
6	$\quad\quad A_1$	5,3 [P]E
7	$\quad A_1$	2,3-6 AMδ
8	$[P]A_0$	2-7 [P]I

1	$\langle F\rangle A_0$	P
2	$\langle F\rangle B_0$	P
3	$[F](A \supset [F]A)_0$	P
4	$[F](B \supset [F]B)_0$	P
5	$\quad 0.1$	A (g, 1$\langle F\rangle$E)
6	$\quad A_1$	
7	$\quad A \supset [F]A_1$	3,5 [F]E
8	$\quad [F]A_1$	7,6 $\supset$E
9	$\quad\quad 0.2$	A (g, 2$\langle F\rangle$E)
10	$\quad\quad B_2$	
11	$\quad\quad B \supset [F]B_2$	2,9 [F]E
12	$\quad\quad [F]B_2$	11,10 $\supset$E
13	$\quad\quad\quad 1.2$	A (g, 5,9 AMφ)
14	$\quad\quad\quad A_2$	8,13 [F]E
15	$\quad\quad\quad A \wedge B_2$	10,14 $\wedge$I
16	$\quad\quad\quad \langle F\rangle(A \wedge B)_0$	9,15 $\langle F\rangle$I
17	$\quad\quad\quad 1 = 2$	A (g, 5,9 AMφ)
18	$\quad\quad\quad A_2$	6,17 =E
19	$\quad\quad\quad A \wedge B_2$	18,10 $\wedge$I
20	$\quad\quad\quad \langle F\rangle(A \wedge B)_0$	9,19 $\langle F\rangle$I
21	$\quad\quad\quad 2.1$	A (g, 5,9 AMφ)
22	$\quad\quad\quad B_1$	12,21 [F]E
23	$\quad\quad\quad A \wedge B_1$	6,22 $\wedge$I
24	$\quad\quad\quad \langle F\rangle(A \wedge B)_0$	5,23 $\langle F\rangle$I
25	$\quad\quad \langle F\rangle(A \wedge B)_0$	5,9 13-16,17-20,21-24 AMφ
26	$\quad \langle F\rangle(A \wedge B)_0$	2,9-25 $\langle F\rangle$E
27	$\langle F\rangle(A \wedge B)_0$	1,5-26 $\langle F\rangle$E

3 Non-Normal Modal Logics: $\boldsymbol{N}\alpha$, $\boldsymbol{L}\alpha$ (ch. 4)

3.1 Language / Semantic Notions

LNα The basic language is the same as for $K\alpha$. The VOCABULARY consists of propositional parameters $p_0, p_1 \ldots$ with the operators, $\neg$, $\wedge$, $\vee$, $\supset$, $\equiv$, $\Box$ and $\Diamond$. Each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$, $(A \equiv B)$, $\Box A$ and $\Diamond A$. In addition, we introduce $(A \prec B)$ as an abbreviation for $\Box(A \supset B)$.

INα An INTERPRETATION is $\langle W, N, R, v\rangle$ where $N \subseteq W$. N is the set of *normal* worlds. Constraints on access are as for $K\alpha$. Thus, where α is empty or indicates some combination of the following constraints,

η	For any x, there is a y such that xRy	extendability
ρ	for all x, xRx	reflexivity
σ	for all x, y, if xRy then yRx	symmetry
τ	for all x, y, z, if xRy and yRz then xRz	transitivity

$\langle W, N, R, v\rangle$ is an $N\alpha$ interpretation when R meets the constraints from α.

TN For complex expressions,

($\neg$) $v_w(\neg A) = 1$ if $v_w(A) = 0$, and 0 otherwise.

($\wedge$) $v_w(A \wedge B) = 1$ if $v_w(A) = 1$ and $v_w(B) = 1$, and 0 otherwise.

($\vee$) $v_w(A \vee B) = 1$ if $v_w(A) = 1$ or $v_w(B) = 1$, and 0 otherwise.

($\supset$) $v_w(A \supset B) = 1$ if $v_w(A) = 0$ or $v_w(B) = 1$, and 0 otherwise.

($\equiv$) $v_w(A \equiv B) = 1$ if $v_w(A) = v_w(B)$, and 0 otherwise.

($\Diamond$) $v_w(\Diamond A) = 1$ if $w \notin N$ or some $x \in W$ such that wRx has $v_x(A) = 1$, and 0 otherwise.

($\Box$) $v_w(\Box A) = 1$ if $w \in N$ and all $x \in W$ such that wRx have $v_x(A) = 1$, and 0 otherwise.

For a set Γ of formulas, $v_w(\Gamma) = 1$ iff $v_w(A) = 1$ for each $A \in \Gamma$; then,

VNα $\Gamma \models_{N\alpha} A$ iff there is no $N\alpha$ interpretation $\langle W, N, R, v\rangle$ and $w \in N$ such that $v_w(\Gamma) = 1$ and $v_w(A) = 0$.

System Lα. An interpretation is $\langle W, N, R, v\rangle$ as before, except that v is a function such that for any $w \in W$, and p, $v_w(p) = 1$ or $v_w(p) = 0$ *and* for any $w \notin N$ and P of the form $\Box A$ or $\Diamond A$, $v_w(P) = 1$ or $v_w(P) = 0$. TN then applies for expressions not assigned a value directly.

3.2 Natural Derivations: *NN*α, *NL*α

All the rules are as in $NK\alpha$ except that whenever a subscript $s.t$ is introduced for $\Box$I or $\Diamond$E, either s is 0 or there is an additional premise of the sort $\Box A_s$, or $\neg\Diamond A_s$ for $NN\alpha$; or s is just 0 for $NL\alpha$. The resulting change on these rules is small.

□I

	$s.t$
	P_t
	$\Box P_s$

where s is 0 for $NL\alpha$; s is 0 or appears in some accessible $\Box A_s$ or $\neg\Diamond A_s$ for $NN\alpha$; and t does not appear in any undischarged premise or assumption

◇E

	$\Diamond P_s$
	$s.t$
	P_t
	Q_u
	Q_u

where s is 0 for $NL\alpha$; s is 0 or appears in some accessible $\Box A_s$ or $\neg\Diamond A_s$ for $NN\alpha$; and t does not appear in any undischarged premise or assumption and is not u

Derived rules carry over from $NK\alpha$. Note that MN remains as well (but restricted to subscript 0 in the L systems). In addition, the following are derived rules for ⥽I and ⥽E in $NK\alpha$, $NN\alpha$ and $NL\alpha$.

⥽I

	$s.t$
	P_t
	Q_t
	$(P ⥽ Q)_s$

constraints on s and t as for the corresponding NL, NN or NK □I rule.

⥽E

	$(P ⥽ Q)_s$
	$s.t$
	P_t
	Q_t

Examples. We exhibit the new restrictions by considering derivations to show one part of MN, that $\Diamond P_s \vdash_{NN\alpha} \neg\Box\neg P_s$. In the case where $s \neq 0$, the derivation on the left violates the NN restriction on ◇E in its last line.

1	$\Diamond P_s$	P
2	$s.t$	A (g, 1 ◇E)
3	P_t	
4	$\Box\neg P_s$	A (c, ¬I)
5	$\neg P_t$	2,4 □E
6	P_t	3 R
7	$\neg\Box\neg P_s$	4-6 ¬I
8	$\neg\Box\neg P_s$	1,2-7 ◇E

1	$\Diamond P_s$	P
2	$\Box\neg P_s$	A (c, ¬I)
3	$s.t$	A (g, 1 ◇E)
4	P_t	
5	$\Box\neg P_s$	A (c, ¬I)
6	$\neg P_t$	3,5 □E
7	P_t	4 R
8	$\neg\Box\neg P_s$	5-7 ¬I
9	$\neg\Box\neg P_s$	2,1,3-8 ◇E
10	$\Box\neg P_s$	2 R
11	$\neg\Box\neg P_s$	2-10 ¬I

Supposing s is 0, each derivation is fine in *NN* and *NL*. However, if s is other than 0, on the left, (8) is automatically bad in *NL* and violates the *NN* restriction on ◇E, insofar as there is no accessible $\Box P_s$ or $\neg\Diamond P_s$. On the right, the derivation works in *NN* even when $s \neq 0$, insofar as we make the assumption for ¬I prior to that for ◇E. Note that, in this case, we *cite* the line with $\Box A_s$ for ◇E. Insofar as $s \neq 0$, the derivation would not do for *NL*.

4 Conditional Logics: Cx (ch. 5)

4.1 Language / Semantic Notions

LCx The VOCABULARY consists of propositional parameters $p_0, p_1 \ldots$ with the operators, $\neg$, $\wedge$, $\vee$, $\supset$, $\equiv$, $\Box$, $\Diamond$ and $>$. Each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$, $(A \equiv B)$, $\Box A$, $\Diamond A$ and $(A > B)$.

ICx Where $\Im$ is the set of all formulas in the language, an INTERPRETATION is $\langle W, \{R_A \mid A \in \Im\}, v\rangle$ where W is a set of worlds, and v assigns 0 or 1 to parameters at worlds. The middle term is a *set* of access relations: for any formula A, there is an access relation R_A which says which worlds are A-accessible from any w. Say $f_A(w) = \{x \in W \,|\, wR_Ax\}$, and $[A] = \{w \,|\, v_w(A) = 1\}$. Then, where x is empty or indicates some combination of the following constraints,

(1) $f_A(w) \subseteq [A]$

(2) If $w \in [A]$, then $w \in f_A(w)$

(3) If $[A] \neq \phi$, then $f_A(w) \neq \phi$

(4) If $f_A(w) \subseteq [B]$ and $f_B(w) \subseteq [A]$, then $f_A(w) = f_B(w)$

(5) If $f_A(w) \cap [B] \neq \phi$, then $f_{A\wedge B}(w) \subseteq f_A(w)$

(6) If $x \in f_A(w)$ and $y \in f_A(w)$, then $x = y$

(7) If $x \in [A]$, and $y \in f_A(x)$, then $x = y$

$\langle W, \{R_A \mid A \in \Im\}, v\rangle$ is a Cx interpretation when it meets the constraints from x. System C has none of the extra constraints; *C+* is C with constraints (1) - (2); *CS* is C with constraints (1) - (5); *C1* is C with constraints (1) - (5) and (7); *C2* is C with constraints (1) - (5) and (6).

TC For complex expressions,

($\neg$) $v_w(\neg A) = 1$ if $v_w(A) = 0$, and 0 otherwise.

($\wedge$) $v_w(A \wedge B) = 1$ if $v_w(A) = 1$ and $v_w(B) = 1$, and 0 otherwise.

($\vee$) $v_w(A \vee B) = 1$ if $v_w(A) = 1$ or $v_w(B) = 1$, and 0 otherwise.

($\supset$) $v_w(A \supset B) = 1$ if $v_w(A) = 0$ or $v_w(B) = 1$, and 0 otherwise.

($\equiv$) $v_w(A \equiv B) = 1$ if $v_w(A) = v_w(B)$, and 0 otherwise.

$(\Diamond)_v$ $v_w(\Diamond A) = 1$ if some $x \in W$ has $v_x(A) = 1$, and 0 otherwise.

$(\Box)_\upsilon$ $v_w(\Box A) = 1$ if all $x \in W$ have $v_x(A) = 1$, and 0 otherwise.

$(>)$ $v_w(A > B) = 1$ iff all $x \in W$ such that wR_Ax have $v_x(B) = 1$.

For a set Γ of formulas, $v_w(\Gamma) = 1$ iff $v_w(A) = 1$ for each $A \in \Gamma$; then,

VCx $\Gamma \models_{Cx} A$ iff there is no Cx interpretation $\langle W, \{R_A \mid A \in \Im\}, v\rangle$ and $w \in W$ such that $v_w(\Gamma) = 1$ and $v_w(A) = 0$.

4.2 Natural Derivations: *NCx*

Derivation systems NCx take over $\neg$, $\supset$, $\wedge$, $\vee$, $\equiv$, $\Box$ and $\Diamond$ rules from $NK\upsilon$. Thus modal rules are,

$$\Box\mathbf{I}\upsilon \quad \begin{array}{|l} \begin{array}{|l} \top_t \\ \hline \\ P_t \end{array} \\ \Box P_s \end{array}$$

where t does not appear in any undischarged premise or assumption

$$\Diamond\mathbf{E}\upsilon \quad \begin{array}{|l} \Diamond P_s \\ \begin{array}{|l} P_t \\ \hline \\ Q_u \end{array} \\ Q_u \end{array}$$

where t does not appear in any undischarged premise or assumption and is not u

$$\Box\mathbf{E}\upsilon \quad \begin{array}{|l} \Box P_s \\ \\ P_t \end{array}$$

$$\Diamond\mathbf{I}\upsilon \quad \begin{array}{|l} P_t \\ \\ \Diamond P_s \end{array}$$

For $>$, let there be new subscripted expressions of the sort $A_{s/t}$ – which intuitively say $w_sR_Aw_t$. Expressions of this sort do not interact with other formulas except as follows (and so do not interact with rules of $NK\upsilon$):

$$>\mathbf{I} \quad \begin{array}{|l} \begin{array}{|l} P_{s/t} \\ \hline \\ Q_t \end{array} \\ (P > Q)_s \end{array}$$

where t does not appear in any undischarged premise or assumption

$$\not>\mathbf{E} \quad \begin{array}{|l} \neg(P > Q)_s \\ \begin{array}{|l} P_{s/t} \\ \neg Q_t \\ \hline \\ R_u \end{array} \\ R_u \end{array}$$

where t does not appear in any undischarged premise or assumption and is not u

$$>\mathbf{E} \quad \begin{array}{|l} (P > Q)_s \\ P_{s/t} \\ \\ Q_t \end{array}$$

$$\not>\mathbf{I} \quad \begin{array}{|l} P_{s/t} \\ \neg Q_t \\ \\ \neg(P > Q)_s \end{array}$$

Corresponding to constraints (1) - (7) are AMp1, AMp2, AMs1, AMs2, AMs3, AMrs, and two forms of AMdl. For AMrs $\mathcal{A}_{(t)}$ is an expression of the sort Q_t, $Q_{t/v}$, $Q_{v/t}$ or $Q_{t/t}$ with a subscript t, and $\mathcal{A}_{(u)}$ is like $\mathcal{A}_{(t)}$ except that some instance(s) of t are replaced by u. And similarly for AMdl.

$$\textbf{AMp1}\ \begin{array}{|l} P_{s/t} \\ \\ P_t \end{array} \qquad \textbf{AMp2}\ \begin{array}{|l} P_t \\ \\ P_{t/t} \end{array} \qquad \textbf{AMs1}\ \begin{array}{|l} \Diamond P_s \\ \begin{array}{|l} P_{s/t} \\ \hline \\ Q_u \end{array} \\ Q_u \end{array} \qquad \textbf{AMs2}\ \begin{array}{|l} (P > Q)_s \\ (Q > P)_s \\ P_{s/t} \\ \\ Q_{s/t} \end{array}$$

where t does not appear in any undischarged premise or assumption and is not u

$$\textbf{AMs3}\ \begin{array}{|l} \neg(P > \neg Q)_s \\ (P \wedge Q)_{s/t} \\ \\ P_{s/t} \end{array} \qquad \textbf{AMrs}\ \begin{array}{|l} P_{s/t} \\ P_{s/u} \\ \mathcal{A}_{(t)} \\ \\ \mathcal{A}_{(u)} \end{array} \qquad \textbf{AMdl}\ \begin{array}{|l} P_s \\ P_{s/t} \\ \mathcal{A}_{(t)} \\ \\ \mathcal{A}_{(s)} \end{array} \quad \begin{array}{|l} P_s \\ P_{s/t} \\ \mathcal{A}_{(s)} \\ \\ \mathcal{A}_{(t)} \end{array}$$

In these systems, every subscript is 0, appears in a premise, or appears in the t-place of an assumption for $\Box$Iv, $\Diamond$Ev, $>$I, $\not>$E or AMs1. Intuitively there are *plus* rules, rules for the *sphere* conception, and rules for the Stalnaker and Lewis alternatives. *NC* includes just the rules of *NK*v plus $>$I, $>$E, $\not>$I and $\not>$E (but, as below, the latter two are derived). Then,

NC+ has the rules of *NC* plus AMp1, AMp2

NCS has the rules of *NC* plus AMp1, AMp2, AMs1, AMs2, AMs3

NC1 has the rules of *NC* plus AMp1, AMp2, AMs1, AMs2, AMs3, AMdl

NC2 has the rules of *NC* plus AMp1, AMp2, AMs1, AMs2, AMs3, AMrs

Derived rules carry over from $NK\alpha$. Where Γ is a set of unsubscripted formulas, let Γ_0 be those same formulas each with subscript 0. Then,

NCx $\Gamma \vdash_{NCx} A$ iff there is an *NCx* derivation of A_0 from Γ_0.

Examples. As first examples, $\not>$I and $\not>$E are derived rules in *NC*, and so in any *NCx*.

$\not>$I

1	$P_{s/t}$	P
2	$\neg Q_t$	P
3	$(P > Q)_s$	A (c, $\neg$I)
4	Q_t	1,3 $>$E
5	$\neg Q_t$	2 R
6	$\neg(P > Q)_s$	3-5 $\neg$I

$\not>$E

1	$\neg(P > Q)_s$	P
2	$\neg R_u$	A (c, $\neg$E)
3	$P_{s/t}$	A (g, $>$I)
4	$\neg Q_t$	A (c, $\neg$E)
	$\vdots$	from 1,3,4
5	R_u	as for $\not>$E
6	$\neg R_u$	2 R
7	Q_t	4-6 $\neg$E
8	$(P > Q)_s$	3-7 $>$I
9	$\neg(P > Q)_s$	1 R
10	R_u	2-9 $\neg$E

As final examples, here is a case in *NCS* using AMs3 and then again in *NC2* but without appeal to AMs3 (so that AMs3 is not necessary in *NC2* for the result). This last case is a bit messy, but should nicely illustrate use of the rules.

$A > B, \neg(A > \neg C) \vdash_{NCS} (A \wedge C) > B$

1	$(A > B)_0$	P
2	$\neg(A > \neg C)_0$	P
3	$(A \wedge C)_{0/1}$	A (g, $>$I)
4	$A_{0/1}$	2,3 AMs3
5	B_1	1,4 $>$E
6	$[(A \wedge C) > B]_0$	3-5 $>$I

$A > B, \neg(A > \neg C) \vdash_{NC2} (A \wedge C) > B$

1	$(A > B)_0$	P
2	$\neg(A > \neg C)_0$	P
3	$A_{0/1}$	A (g, 2 $\not>$E)
4	$\neg\neg C_1$	
5	$(A \wedge C)_{0/2}$	A (g, $>$I)
6	$(A \wedge C)_{0/3}$	A (g, $>$I)
7	$(A \wedge C)_3$	6 AMp1
8	A_3	7 $\wedge$E
9	$[(A \wedge C) > A]_0$	6-8 $>$I
10	$A_{0/3}$	A (g, $>$I)
11	A_3	10 AMp1
12	$\neg\neg C_3$	3,10,4 AMrs
13	C_3	12 DN
14	$(A \wedge C)_3$	11,13 $\wedge$I
15	$[A > (A \wedge C)]_0$	10-14 $>$I
16	$A_{0/2}$	9,15,5 AMs2
17	B_2	1,16 $>$E
18	$[(A \wedge C) > B]_0$	5-17 $>$I
19	$[(A \wedge C) > B]_0$	2,3-18 $\not>$E

The derivation on the left is a simple application of AMs3. On the right, we go for the final goal by $\not>$E.[2] The real work is getting $A_{0/2}$ so that we can use $>$E with (1). And we go for this by getting the conditionals that feed into AMs2, given that we already have $(A \wedge C)_{0/2}$.

5 Intuitionistic Logic: *IL* (ch. 6)

5.1 Language / Semantic Notions

LIL The VOCABULARY consists of propositional parameters $p_0, p_1 \ldots$ with the operators, $\wedge$, $\vee$, $\neg$, and $\sqsupset$. Each propositional parameter is a FORMULA; if A and B are formulas, so are $(A \wedge B)$, $(A \vee B)$, $\neg A$, and $(A \sqsupset B)$.

IIL An INTERPRETATION is a triple $\langle W, R, v\rangle$ where W is a set of worlds, R is a subset of $W^2 = W \times W$, and v is a function such that for any $w \in W$ and p, $v_w(p) = 1$ or $v_w(p) = 0$. For $x, y, z \in W$, an interpretation is subject to the conditions,

ρ	for all x, xRx	reflexivity
τ	for all x, y, z, if xRy and yRz then xRz	transitivity
h	for any parameter p, if $v_x(p) = 1$, and xRy, then $v_y(p) = 1$	heredity

We think of worlds as representing a state of information at a given time. $v_w(p) = 1$ when p is proved at state w. The heredity condition guarantees that what is proved at one stage remains proved at the next. Notice that $v_w(p) = 0$ does not indicate that p is *false* – but rather that p *isn't proved*.

TIL For complex expressions,

($\wedge$) $v_w(A \wedge B) = 1$ if $v_w(A) = 1$ and $v_w(B) = 1$, and 0 otherwise.

($\vee$) $v_w(A \vee B) = 1$ if $v_w(A) = 1$ or $v_w(B) = 1$, and 0 otherwise.

($\neg$) $v_w(\neg A) = 1$ if all $x \in W$ such that wRx have $v_x(A) = 0$, and 0 otherwise.

($\sqsupset$) $v_w(A \sqsupset B) = 1$ if all $x \in W$ such that wRx have either $v_x(A) = 0$ or $v_x(B) = 1$, and 0 otherwise.

For a set Γ of formulas, $v_w(\Gamma) = 1$ iff $v_w(A) = 1$ for each $A \in \Gamma$; then,

[2] As, given strategies from [6, chapter 6], we would jump on $\vee$E, $\exists$E or $\Diamond$E when available.

VIL $\Gamma \models_{IL} A$ iff there is no IL interpretation $\langle W, R, v\rangle$ and $w \in W$ such that $v_w(\Gamma) = 1$ and $v_w(A) = 0$.

5.2 Natural Derivations: *NIL*

Augment the language for intuionistic logic to include expressions with subscripts and expressions of the sort $s.t$ as for $NK\alpha$, along with a unary operator, $\sim$. Intuitively, $\sim A$ indicates that A is not (yet) proven. There is one new rule for the heredity condition. Otherwise, rules are as in $NK\rho\tau$ with $\sim$ like $\neg$, and rules for $\sqsupset$ and $\rightharpoondown$ on the analogy of ⥽ and $\Box\neg$.

$$\textbf{R}\ \begin{array}{|l} P_s \\ \\ P_s \end{array} \qquad \textbf{H}\ \begin{array}{|l} P_s \\ s.t \\ \\ P_t \end{array}$$

where P includes no instance of $\sim$

$$\wedge\textbf{I}\ \begin{array}{|l} P_s \\ Q_s \\ \\ (P \wedge Q)_s \end{array} \qquad \wedge\textbf{E}\ \begin{array}{|l} (P \wedge Q)_s \\ \\ P_s \end{array} \qquad \wedge\textbf{E}\ \begin{array}{|l} (P \wedge Q)_s \\ \\ Q_s \end{array}$$

$$\vee\textbf{I}\ \begin{array}{|l} P_s \\ \\ (P \vee Q)_s \end{array} \qquad \vee\textbf{I}\ \begin{array}{|l} P_s \\ \\ (Q \vee P)_s \end{array} \qquad \vee\textbf{E}\ \begin{array}{|l} (P \vee Q)_s \\ \begin{array}{|l} \underline{P_s} \\ \\ R_t \end{array} \\ \begin{array}{|l} \underline{Q_s} \\ \\ R_t \end{array} \\ R_t \end{array}$$

$$\sim\textbf{I}\ \begin{array}{|l} \begin{array}{|l} \underline{P_s} \\ \\ Q_t \\ \sim Q_t \end{array} \\ \sim P_s \end{array} \qquad \sim\textbf{E}\ \begin{array}{|l} \begin{array}{|l} \underline{\sim P_s} \\ \\ Q_t \\ \sim Q_t \end{array} \\ P_s \end{array}$$

$$\sqsupset\textbf{I}\ \begin{array}{|l} \begin{array}{|l} s.t \\ \underline{P_t} \\ \\ Q_t \end{array} \\ (P \sqsupset Q)_s \end{array} \qquad \sqsupset\textbf{E}\ \begin{array}{|l} (P \sqsupset Q)_s \\ s.t \\ P_t \\ \\ Q_t \end{array} \qquad \textbf{AM}\rho\ \begin{array}{|l} \\ \\ s.s \end{array}$$

where t does not appear in any undischarged premise or assumption

$$\rightharpoondown\textbf{I}\ \begin{array}{|l} \begin{array}{|l} \underline{s.t} \\ \\ \sim P_t \end{array} \\ \rightharpoondown P_s \end{array} \qquad \rightharpoondown\textbf{E}\ \begin{array}{|l} \rightharpoondown P_s \\ s.t \\ \\ \sim P_t \end{array} \qquad \textbf{AM}\tau\ \begin{array}{|l} s.t \\ t.u \\ \\ s.u \end{array}$$

where t does not appear in any undischarged premise or assumption

Every subscript is 0, appears in a premise, or appears in the t-place of an accessible assumption for $\sqsupset$I or $\neg$I. Where the members of Γ and A are formulas in the original language for intuitionistic logic (without subscripts and without $\sim$), let let the members of Γ_0 be the formulas in Γ, each with subscript 0. Then,

NIL $\Gamma \vdash_{NIL} A$ iff there is an *NIL* derivation of A_0 from the members of Γ_0.

Examples. Here are instances of the more interesting standard axioms for intuitionistic logic. Note that our account of a derivation guarantees that $\sim$ is not an operator in any of A, B, or C.

A1 $\vdash_{NIL} A \sqsupset (B \sqsupset A)$

1	0.1	A (g, $\sqsupset$I)
2	A_1	
3	1.2	A (g, $\sqsupset$I)
4	B_2	
5	A_2	2,3 H
6	$(B \sqsupset A)_1$	3-5 $\sqsupset$I
7	$[A \sqsupset (B \sqsupset A)]_0$	1-6 $\sqsupset$I

A2 $\vdash_{NIL} (A \sqsupset B) \sqsupset [(A \sqsupset (B \sqsupset C)) \sqsupset (A \sqsupset C)]$

1	0.1	A (g, $\sqsupset$I)
2	$(A \sqsupset B)_1$	
3	1.2	A (g, $\sqsupset$I)
4	$(A \sqsupset (B \sqsupset C))_2$	
5	2.3	A (g, $\sqsupset$I)
6	A_3	
7	1.3	3,5 AMτ
8	B_3	2,7,6 $\sqsupset$E
9	$(B \sqsupset C)_3$	4,5,6 $\sqsupset$E
10	3.3	AMρ
11	C_3	9,10,8 $\sqsupset$E
12	$(A \sqsupset C)_2$	5-11 $\sqsupset$I
13	$[(A \sqsupset (B \sqsupset C)) \sqsupset (A \sqsupset C)]_1$	3-12 $\sqsupset$I
14	$((A \sqsupset B) \sqsupset [(A \sqsupset (B \sqsupset C)) \sqsupset (A \sqsupset C)])_0$	1-13 $\sqsupset$I

A3 $\vdash_{NIL} A \sqsupset (B \sqsupset (A \wedge B))$

A4 $\vdash_{NIL} (A \wedge B) \sqsupset A$

A5 $\vdash_{NIL} (A \wedge B) \sqsupset B$

A6 $\vdash_{NIL} A \sqsupset (A \vee B)$

A7 $\vdash_{NIL} B \sqsupset (A \vee B)$

A8 $\vdash_{NIL} (A \sqsupset C) \sqsupset [(B \sqsupset C) \sqsupset ((A \vee B) \sqsupset C)]$

A9 $\vdash_{NIL} (A \sqsupset B) \sqsupset [(A \sqsupset \neg B) \sqsupset \neg A]$

1	0.1	A (g, $\sqsupset$I)
2	$(A \sqsupset B)_1$	
3	1.2	A (g, $\sqsupset$I)
4	$(A \sqsupset \neg B)_2$	
5	2.3	A (g, $\neg$I)
6	A_3	A (c, $\sim$I)
7	1.3	3,5 AMτ
8	B_3	2,7,6 $\sqsupset$E
9	$\neg B_3$	4,5,6 $\sqsupset$E
10	3.3	AMρ
11	$\sim B_3$	9,10 $\neg$E
12	$\sim A_3$	6-11 $\sim$I
13	$\neg A_2$	5-12 $\neg$I
14	$[(A \sqsupset \neg B) \sqsupset \neg A]_1$	3-13 $\sqsupset$I
15	$((A \sqsupset B) \sqsupset [(A \sqsupset \neg B) \sqsupset \neg A])_0$	1-14 $\sqsupset$I

A10 $\vdash_{NIL} \neg A \sqsupset (A \sqsupset B)$

1	0.1	A (g, $\sqsupset$I)
2	$\neg A_1$	
3	1.2	A (g, $\sqsupset$I)
4	A_2	
5	$\sim B_2$	A (c, $\sim$E)
6	A_2	4 R
7	$\sim A_2$	2,3 $\neg$E
8	B_2	5-7 $\sim$E
9	$(A \sqsupset B)_1$	3-8 $\sqsupset$I
10	$[\neg A \sqsupset (A \sqsupset B)]_0$	1-9 $\sqsupset$I

A system with these axioms and MP (which we already have by AMρ with $\sqsupset$E) turns into classical logic if A10 is replaced by double negation, $\neg\neg A \sqsupset A$. But we cannot prove $\neg\neg A \sqsupset A$ (or at least we cannot if our derivation system is sound).

6 Many-Valued Logics: *Mx* (ch. 7,8)

6.1 Language / Semantic Notions

LMx The LANGUAGE consists of propositional parameters $p_0, p_1 \ldots$ with the operators, $\neg$, $\wedge$, $\vee$, and $\supset$. Each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, and $(A \supset B)$. $A \equiv B$ abbreviates $(A \supset B) \wedge (B \supset A)$.

IMx An INTERPRETATION is a function v which assigns to each propositional parameter some subset of $\{0, 1\}$; so $v(p)$ is ϕ, $\{1\}$, $\{0\}$ or $\{1, 0\}$. Intuitively, $v(p)$ is true iff $1 \in v(p)$ and $v(p)$ is false iff $0 \in v(p)$. Where x is empty or includes some combination of the following constraints,

exc	for no p are both $0 \in v(p)$ and $1 \in v(p)$	exclusion
exh	for any p, either $1 \in v(p)$ or $0 \in v(p)$	exhaustion

v is an *Mx* interpretation only if it meets the constraints from x. *MCL* has both *exc* and *exh*, *MK3* and *MŁ3* just *exc*, *MLP* and *MRM* just *exh*, and *MFD* neither *exc* nor *exh* (these are classical logic, and Priest's *K3*, *Ł3*, *LP*, *RM3* and *FDE*).

TM For complex expressions,

($\neg$) $1 \in v(\neg A)$ iff $0 \in v(A)$; $0 \in v(\neg A)$ iff $1 \in v(A)$.

($\wedge$) $1 \in v(A \wedge B)$ iff $1 \in v(A)$ and $1 \in v(B)$; $0 \in v(A \wedge B)$ iff $0 \in v(A)$ or $0 \in v(B)$.

($\vee$) $1 \in v(A \vee B)$ iff $1 \in v(A)$ or $1 \in v(B)$; $0 \in v(A \vee B)$ iff $0 \in v(A)$ and $0 \in v(B)$.

($\supset$) $1 \in v(A \supset B)$ iff $0 \in v(A)$ or $1 \in v(B)$; $0 \in v(A \supset B)$ iff $1 \in v(A)$ and $0 \in v(B)$.

$(\supset)_{L3}$ $1 \in v(A \supset B)$ iff $0 \in v(A)$ or $1 \in v(B)$ or none of $1, 0 \in v(A)$ or $1, 0 \in v(B)$; $0 \in v(A \supset B)$ iff $1 \in v(A)$ and $0 \in v(B)$.

$(\supset)_{RM}$ $1 \in v(A \supset B)$ iff $1 \notin v(A)$ or $0 \notin v(B)$ or all of $1, 0 \in v(A)$ and $1, 0 \in v(B)$; $0 \in v(A \supset B)$ iff $1 \in v(A)$ and $0 \in v(B)$.

All the systems have the same conditions, except that *MŁ3* interpretations use $(\supset)_{L3}$ and *MRM* interpretations use $(\supset)_{RM}$. For a set Γ of formulas, $1 \in v(\Gamma)$ iff $1 \in v(A)$ for each $A \in \Gamma$; then,

VMx $\Gamma \models_{Mx} A$ iff there is no *Mx* interpretation v such that $v(\Gamma) = 1$ but $1 \notin v(A)$.

This account is adequate to the (superficially) different presentations in these chapters of Priest. For the multivalued approach: classical logic has values $\{0\}$, $\{1\}$, with $\{1\}$ designated; *K3* and *Ł3* have ϕ, $\{0\}$, $\{1\}$, with $\{1\}$ designated; *LP* and *RM3* have $\{0\}$, $\{1\}$, $\{0,1\}$, with $\{1\}$ and $\{0,1\}$ designated; and *FDE* has ϕ, $\{0\}$, $\{1\}$, $\{0,1\}$, with $\{1\}$ and $\{0,1\}$ designated. For the relational approach, we identify the relation as set membership. And a v as above maps to a Routley interpretation with $v_w(p) = 1$ iff $1 \in v(p)$, and $v_{w^*}(p) = 0$ iff $0 \in v(p)$.[3] Then, in each case, conditions for truth and validity are as above.

6.2 Natural Derivations: *NMx*

Introduce expressions of the sort A and $\overline{A}$. Intuitively $\overline{A}$ indicates that A is *not false*. Let $\backslash A\backslash$ and $/A/$ represent either A or $\overline{A}$ where what is represented is constant in a given context, but $\backslash A\backslash$ and $/A/$ are opposite. And similarly for $/\!/A/\!/$ and $\backslash\!\backslash A\backslash\!\backslash$, though there need be no fixed relation between overlines on $\backslash A\backslash$ and $\backslash\!\backslash A\backslash\!\backslash$. Except for a pair of new rules (D) and (U) corresponding to conditions *exc* and *exh*, derivation rules mirror ones for classical logic.

D | P
  | $\overline{P}$

U | $\overline{P}$
  | P

R | $/P/$
  | $/P/$

¬I | | $/P/$ (assumption)
  | | $/\!/Q/\!/$
  | | $\backslash\!\backslash\neg Q\backslash\!\backslash$
  | $\backslash\neg P\backslash$

¬E | | $/\neg P/$ (assumption)
  | | $/\!/Q/\!/$
  | | $\backslash\!\backslash\neg Q\backslash\!\backslash$
  | $\backslash P\backslash$

∧I | $/P/$
  | $/Q/$
  | $/P \wedge Q/$

∧E | $/P \wedge Q/$
  | $/P/$

∧E | $/P \wedge Q/$
  | $/Q/$

[3] For this, see [3, sections 8.5.8, 8.7.17 and 8.7.18] along with L6.0 for the proof of soundness in [7].

$$\vee\mathbf{I}\ \begin{array}{|l} /P/ \\ \\ /P \vee Q/ \end{array} \qquad \vee\mathbf{I}\ \begin{array}{|l} /P/ \\ \\ /Q \vee P/ \end{array} \qquad \vee\mathbf{E}\ \begin{array}{|l} /P \vee Q/ \\ \begin{array}{|l} /P/ \\ \hline \\ /\!/R/\!/ \end{array} \\ \begin{array}{|l} /Q/ \\ \hline \\ /\!/R/\!/ \end{array} \\ /\!/R/\!/ \end{array}$$

$$\supset\mathbf{I}\ \begin{array}{|l} \begin{array}{|l} /P/ \\ \hline \\ \backslash Q \backslash \end{array} \\ \backslash P \supset Q \backslash \end{array} \qquad \supset\mathbf{E}\ \begin{array}{|l} \backslash P \supset Q \backslash \\ /P/ \\ \\ \backslash Q \backslash \end{array}$$

$$\equiv\mathbf{I}\ \begin{array}{|l} \begin{array}{|l} /P/ \\ \hline \\ \backslash Q \backslash \end{array} \\ \begin{array}{|l} /Q/ \\ \hline \\ \backslash P \backslash \end{array} \\ \backslash P \equiv Q \backslash \end{array} \qquad \equiv\mathbf{E}\ \begin{array}{|l} \backslash P \equiv Q \backslash \\ /P/ \\ \\ \backslash Q \backslash \end{array} \qquad \equiv\mathbf{E}\ \begin{array}{|l} \backslash P \equiv Q \backslash \\ /Q/ \\ \\ \backslash P \backslash \end{array}$$

NMFD has the I- and E-rules for $\neg$, $\wedge$, $\vee$, $\supset$ with (R). *NMK3* adds (D), for truth *down*. *NMLP* adds (U), for truth *up*. *NMCL* has all the rules. In these systems, ($\equiv$I) and ($\equiv$E) are derived. In addition, for these systems, two-way derived rules carry over from *CL* with consistent overlines. Thus, e.g.,

$$\mathbf{Impl} \quad \begin{array}{l} /P \supset Q/ \ \triangleleft\triangleright\ /\neg P \vee Q/ \\ /\neg P \supset Q/ \ \triangleleft\triangleright\ /P \vee Q/ \end{array}$$

MT, NB and DS appear in the forms,

$$\mathbf{MT}\ \begin{array}{|l} /P \supset Q/ \\ \backslash\neg Q\backslash \\ \\ /\neg P/ \end{array} \qquad \mathbf{NB}\ \begin{array}{|l} /P \equiv Q/ \\ \backslash\neg P\backslash \\ \\ /\neg Q/ \end{array} \quad \begin{array}{|l} /P \equiv Q/ \\ \backslash\neg Q\backslash \\ \\ /\neg P/ \end{array} \qquad \mathbf{DS}\ \begin{array}{|l} /P \vee Q/ \\ \backslash\neg P\backslash \\ \\ /Q/ \end{array} \quad \begin{array}{|l} /P \vee Q/ \\ \backslash\neg Q\backslash \\ \\ /P/ \end{array}$$

Alternate systems. The systems *NMŁ3* and *NMRM* have (R) with I and E rules for $\neg$, $\wedge$, and $\vee$. Both include,

$$\overline{\supset}\mathrm{I}\ \begin{array}{|l} \begin{array}{|l} P \\ \hline \\ \overline{Q} \end{array} \\ \overline{P \supset Q} \end{array} \qquad \overline{\supset}\mathrm{E}\ \begin{array}{|l} \overline{P \supset Q} \\ P \\ \\ \overline{Q} \end{array}$$

which are the same as before. *NMŁ3* adds (D) and,

$\supset\mathrm{I}_{L3}$

$$\begin{array}{l|l} & \left|\begin{array}{l} \overline{P} \\ \underline{(P \vee \neg P) \vee (Q \vee \neg Q)} \\ \\ Q \end{array}\right. \\ & P \supset Q \end{array}$$

$\supset\mathrm{E}_{L3}$

$$\begin{array}{|l} P \supset Q \\ \underline{(P \vee \neg P) \vee (Q \vee \neg Q)} \\ \overline{P} \\ \\ Q \end{array}$$

NM_{RM} adds (U) and,

$\supset\mathrm{I}_{RM}$

$$\begin{array}{l|l} & \left|\begin{array}{l} \underline{P \qquad\qquad\qquad\qquad} \\ \underline{(P \vee \neg P) \vee (Q \vee \neg Q)} \\ \\ \overline{Q} \end{array}\right. \\ & P \supset Q \end{array}$$

$\supset\mathrm{E}_{RM}$

$$\begin{array}{|l} \underline{P \supset Q \qquad\qquad\qquad} \\ (P \vee \neg P) \vee (Q \vee \neg Q) \\ P \\ \\ \overline{Q} \end{array}$$

Because of the lack of symmetry for $\supset$ rules, there is no easy carryover in these systems of derived rules for $\equiv$ and $\supset$.

Where the members of Γ and A are expressions without overlines,

NMx $\Gamma \vdash_{NMx} A$ iff there is an *NMx* derivation of A from the members of Γ.

Examples. Here are derivations, cast to show the general forms, for MT and the second form of DS.

$/P \supset Q/, \backslash\neg Q\backslash \vdash_{NMx} /\neg P/$

1	$/P \supset Q/$	P
2	$\backslash\neg Q\backslash$	P
3	$\quad \backslash P\backslash$	A (c, $\neg$I)
4	$\quad /Q/$	1,3 $\supset$E
5	$\quad \backslash\neg Q\backslash$	2 R
6	$/\neg P/$	3-5 $\neg$I

$/P \vee Q/, \backslash\neg Q\backslash \vdash_{NMx} /P/$

1	$/P \vee Q/$	P
2	$\backslash\neg Q\backslash$	P
3	$\quad /P/$	A (g, 1 $\vee$E)
4	$\quad /P/$	3 R
5	$\quad /Q/$	A (g, 1 $\vee$E)
6	$\quad\quad \backslash\neg P\backslash$	A (c, $\neg$E)
7	$\quad\quad /Q/$	5 R
8	$\quad\quad \backslash\neg Q\backslash$	2 R
9	$\quad /P/$	6-8 $\neg$E
10	$/P/$	1,3-4,5-9 $\vee$E

And for some particular results requiring (D) and (U), here are demonstrations of standard rule and axioms for classical logic, making use of the full rule set (see, e.g. [6, chapter 3]).

MP $A, A \supset B \vdash_{NM_{CL}} B$

1	A	P
2	$A \supset B$	P
3	$\overline{A}$	1 D
4	B	2,3 ⊃E

A1 $\vdash_{NM_{CL}} A \supset (B \supset A)$

1	$\mid\ \overline{A}$	A (g, ⊃I)
2	$\mid\ \mid\ \overline{B}$	A (g, ⊃I)
3	$\mid\ \mid\ A$	1 U
4	$\mid\ B \supset A$	2-3 ⊃I
5	$A \supset (B \supset A)$	1-4 ⊃I

A2 $\vdash_{NM_{CL}} [A \supset (B \supset C)] \supset [(A \supset B) \supset (A \supset C)]$

1	$\mid\ \overline{A \supset (B \supset C)}$	A (g, ⊃I)
2	$\mid\ \mid\ \overline{A \supset B}$	A (g, ⊃I)
3	$\mid\ \mid\ \mid\ \overline{A}$	A (g, ⊃I)
4	$\mid\ \mid\ \mid\ A \supset B$	2 U
5	$\mid\ \mid\ \mid\ B$	3,4 ⊃E
6	$\mid\ \mid\ \mid\ A \supset (B \supset C)$	1 U
7	$\mid\ \mid\ \mid\ B \supset C$	3,6 ⊃E
8	$\mid\ \mid\ \mid\ \overline{B}$	5 D
9	$\mid\ \mid\ \mid\ C$	7,8 ⊃E
10	$\mid\ \mid\ A \supset C$	3-9 ⊃I
11	$\mid\ (A \supset B) \supset (A \supset C)$	2-10 ⊃I
12	$[A \supset (B \supset C)] \supset [(A \supset B) \supset (A \supset C)]$	1-11 ⊃I

A3 $\vdash_{NM_{CL}} (\neg A \supset \neg B) \supset [(\neg A \supset B) \supset A]$

1	$\mid\ \overline{\neg A \supset \neg B}$	A (g, ⊃I)
2	$\mid\ \mid\ \overline{\neg A \supset B}$	A (g, ⊃I)
3	$\mid\ \mid\ \mid\ \overline{\neg A}$	A (c, ¬E)
4	$\mid\ \mid\ \mid\ \neg A$	3 U
5	$\mid\ \mid\ \mid\ \overline{B}$	2,4 ⊃E
6	$\mid\ \mid\ \mid\ \overline{\neg B}$	1,4 ⊃E
7	$\mid\ \mid\ \mid\ \neg B$	6 U
8	$\mid\ \mid\ A$	3-7 ¬E
9	$\mid\ (\neg A \supset B) \supset A$	2-8 ⊃I
10	$(\neg A \supset \neg B) \supset [(\neg A \supset B) \supset A]$	1-9 ⊃I

Of course, there is not much point going back-and-forth between overline and non-overline expressions in the full classical system. But these examples should illustrate the rules. And overlines matter for the other systems. Finally, a couple derivations to show *modus ponens* as a derived rule in *NML3* and *NMRM*.

$P \supset Q, P \vdash_{NM_{L3}} Q$

1	$P \supset Q$	P
2	P	P
3	$\overline{P}$	2 D
4	$P \vee \neg P$	2 ∨I
5	$(P \vee \neg P) \vee (Q \vee \neg Q)$	4 ∨I
6	Q	1,3,5 ⊃E

$P \supset Q, P \vdash_{NM_{RM}} Q$

1	$P \supset Q$	P
2	P	P
3	$\quad \mid \overline{\neg Q}$	A (c, ¬E)
4	$\quad \mid \overline{Q \vee \neg Q}$	3 ∨I
5	$\quad \mid \overline{(P \vee \neg P) \vee (Q \vee \neg Q)}$	4 ∨I
6	$\quad \mid \overline{Q}$	1,2,5 ⊃E
7	$\quad \mid \neg Q$	3 U
8	Q	3-7 ¬E

7 Gaps, Gluts and Worlds: $\upsilon \boldsymbol{X}$, $\boldsymbol{Ix}$ (ch. 9)

7.1 Language / Semantic Notions

This section is developed directly in terms introduced in demonstration of soundness and completeness in section 6. Apart from that discussion, the notions should be roughly familiar from derivations in that section.

LυX The VOCABULARY consists of propositional parameters $p_0, p_1 \ldots$ with the operators, $\neg$, $\wedge$, $\vee$, and $\rightarrow$. Each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, and $(A \rightarrow B)$. $A \supset B$ abbreviates $\neg A \vee B$, and $A \equiv B$ abbreviates $(A \supset B) \wedge (B \supset A)$. This time, from the start, for any formula A, we allow A and $\overline{A}$, where as before $/A/$ and $\backslash A \backslash$ ($/\!\!/ A /\!\!/$ and $\backslash\!\!\backslash A \backslash\!\!\backslash$) represent one or the other (and similarly for N and $\overline{N}$ immediately below).

IυX An INTERPRETATION is $\langle W, N, \overline{N}, h \rangle$ where W is a set of worlds, and $N, \overline{N} \subseteq W$ are normal worlds for truth and non-falsity respectively; h is a function such that for any $w \in W$, $h_w(/p/) = 1$ or $h_w(/p/) = 0$, and for any w not in $/N/$, $h_w(/A \rightarrow B/) = 1$ or $h_w(/A \rightarrow B/) = 0$. So

h makes assignments directly to experssions of the sort $/A \to B/$ at worlds not in $/N/$. Say $/A/$ *holds* at w if $h_w(/A/) = 1$ and otherwise *fails*. Interpretations may also be subject to the constraints,

K	$N = \overline{N} = W$
4	$N = \overline{N}$

The K systems are subject to constraint (K), the 4 systems to (4). Of course, (K) implies (4); so it is enough that interpretations for υK_4 and υK_* are subject to (K); υN_4 is subject to (4), and υN_* to neither. With restriction K, h reduces to a simple assignment to parameters at worlds. Though it does not appear in Priest, we consider also a requirement (CL) which includes (4) and that for any for any $w \in N$, $h_w(p) = h_w(\overline{p})$.

Hυ For expressions not assigned a value directly,

- ($\neg$) $h_w(/\neg A/) = 1$ if $h_w(\backslash A\backslash) = 0$, and 0 otherwise.
- ($\wedge$) $h_w(/A \wedge B/) = 1$ if $h_w(/A/) = 1$ and $h_w(/B/) = 1$, and 0 otherwise.
- ($\vee$) $h_w(/A \vee B/) = 1$ if $h_w(/A/) = 1$ or $h_w(/B/) = 1$, and 0 otherwise.
- $(\to)_4$ For $w \in /N/$, $h_w(/A \to B/) = 1$ iff there is no $x \in W$ such that $h_x(A) = 1$ and $h_x(/B/) = 0$.
- $(\to)_*$ For $w \in /N/$, $h_w(/A \to B/) = 1$ iff there is no $x \in W$ such that $h_x(/\!\!/A/\!\!/) = 1$ and $h_x(/\!\!/B/\!\!/) = 0$.

The 4-systems υN_4 and υK_4 take H$\upsilon(\to)_4$; the star systems υN_* and υK_* take H$\upsilon(\to)_*$. Where Γ does not include formulas with overlines, $h_w(\Gamma) = 1$ iff $h_w(A) = 1$ for each $A \in \Gamma$; then,

VυX $\Gamma \models_{\upsilon X} A$ iff there is no υX interpretation $\langle W, N, \overline{N}, h\rangle$ and $w \in N$ such that $h_w(\Gamma) = 1$ and $h_w(A) = 0$.

System Ix.

LIx The vocabulary is as before with $\sqsupset$ for $\to$. Again, for any formula A, allow A and $\overline{A}$.

IIx An interpretation is $\langle W, R, h\rangle$ where,

ρ	for all x, xRx	reflexivity
τ	for all x, y, z, if xRy and yRz then xRz	transitivity
h	for all x, y and p, if xRy, then if $h_x(p) = 1$, $h_y(p) = 1$, and if $h_y(\overline{p}) = 1$, $h_x(\overline{p}) = 1$	heredity

apply to any interpretation. In addition, interpretations may be subject to the condition,

exc	for no p is both $h(p) = 1$ and $h(\overline{p}) = 0$	exclusion

HIx is as before with,

($\sqsupset$) $h_x(A \sqsupset B) = 1$ iff there is no $y \in W$ such that xRy and $h_y(A) = 1$ but $h_y(B) = 0$. $h_x(\overline{A \sqsupset B}) = 1$ iff $h_x(A) = 0$ or $h_y(\overline{B}) = 1$.

$(\sqsupset)_W$ $h_x(A \sqsupset B) = 1$ iff there is no $y \in W$ such that xRy and $h_y(A) = 1$ but $h_y(B) = 0$. $h_x(\overline{A \sqsupset B}) = 1$ iff there is some $y \in W$ such that xRy and $h_y(A) = 1$ and $h_y(\overline{B}) = 1$.

The system I_4 takes neither *exc* nor $(\sqsupset)_W$. I_3 adds *exc*; $I W$ adds to I_4 the $(\sqsupset)_W$ condition. Then validity works in the usual way.

As in the previous section, these accounts are meant to accommodate different presentations in Priest, and help exhibit their differences. In particular, as for the previous section, given constraint (4), an interpretation $\langle W, N, \overline{N}, h\rangle$ corresponds to a relational $\langle W, N, \rho\rangle$, where $h_w(A) = 1$ iff A bears relation ρ (which, as in the previous section, may be set membership) to 1 at w, and $h_w(\overline{A}) = 1$ iff A does not bear ρ to 0 at w. And an interpretation $\langle W, N, \overline{N}, h\rangle$ corresponds to a star interpretation $\langle W, N, *, v\rangle$ where $h_w(A) = 1$ iff $v_w(A) = 1$ and $h_w(\overline{A}) = 1$ iff $v_{w^*}(A) = 1$.[4]

7.2 Natural Derivations: $N\upsilon X$, NIx

Allow expressions with both integer subscripts and overlines. $/n/[s]$ indicates that world s is an element of $/N/$. I- and E- rules for $\neg$, $\wedge$, $\vee$, $\supset$ and $\equiv$ are

[4]For the latter, given a star interpretation $\langle W, N, *, v\rangle$ consider an υX_* interpretation $\langle W', N', \overline{N}', h\rangle$ with a $w' \in W'$ corresponding to each $w \in W$. And for an υX_* interpretation $\langle W', N', \overline{N}', h\rangle$ consider a star interpretation $\langle W, N, *, v\rangle$ with a w and $w^* \in W$ corresponding to each $w' \in W'$. Then set $x' \in N'$ iff $x \in N$; $x' \in \overline{N}'$ iff $x^* \in N$; $h_{x'}(p) = 1$ iff $v_x(p) = 1$; $h_{x'}(\overline{p}) = 1$ iff $v_{x^*}(p) = 1$; for $x' \notin N'$, $h_{x'}(P \to Q) = 1$ iff $v_x(P \to Q) = 1$; and for $x' \notin \overline{N}'$, $h_{x'}(\overline{P \to Q}) = 1$ iff $v_{x^*}(P \to Q) = 1$. Then the result follows by a simple induction (for a related demonstration, see the proof of L7.0 in [7]).

a natural combination of rules for *NKv* and *NFDE*, with rules for $\supset$ and $\equiv$ now derived.

R
$/P/_s$

$/P/_s$

¬I
| $/P/_s$
|
| $/\!/Q/\!/_t$
| $\backslash\!\backslash\neg Q\backslash\!\backslash_t$
$\backslash\neg P\backslash_s$

¬E
| $/\neg P/_s$
|
| $/\!/Q/\!/_t$
| $\backslash\!\backslash\neg Q\backslash\!\backslash_t$
$\backslash P\backslash_s$

∧I
$/P/_s$
$/Q/_s$

$/P \wedge Q/_s$

∧E
$/P \wedge Q/_s$

$/P/_s$

∧E
$/P \wedge Q/_s$

$/Q/_s$

∨I
$/P/_s$

$/P \vee Q/_s$

∨I
$/P/_s$

$/Q \vee P/_s$

∨E
$/P \vee Q/_s$
| $/P/_s$
|
| $/\!/R/\!/_t$

| $/Q/_s$
|
| $/\!/R/\!/_t$
$/\!/R/\!/_t$

⊃I
| $/P/_s$
|
| $\backslash Q\backslash_s$
$\backslash P \supset Q\backslash_s$

⊃E
$\backslash P \supset Q\backslash_s$
$/P/_s$

$\backslash Q\backslash_s$

≡I
| $/P/_s$
|
| $\backslash Q\backslash_s$

| $/Q/_s$
|
| $\backslash P\backslash_s$
$\backslash P \equiv Q\backslash_s$

≡E
$\backslash P \equiv Q\backslash_s$
$/P/_s$

$\backslash Q\backslash_s$

≡E
$\backslash P \equiv Q\backslash_s$
$/Q/_s$

$\backslash P\backslash_s$

The different derivation systems of this section add to these from,

→I4
$/n/[s]$
| P_t
|
| $/P/_t$
$/P \rightarrow Q/_s$

where t does not appear in any undischarged premise or assumption

→E4
$/n/[s]$
$/P \rightarrow Q/_s$
P_t

$/Q/_t$

→I∗
$/n/[s]$
| $/\!/P/\!/_t$
|
| $/\!/Q/\!/_t$
$/P \rightarrow Q/_s$

where t does not appear in any undischarged premise or assumption

→E∗
$/n/[s]$
$/P \rightarrow Q/_s$
$/\!/P/\!/_t$

$/\!/Q/\!/_t$

$$\begin{array}{r|l} \textbf{K} & \\ & \\ & /n/[s] \end{array} \qquad \begin{array}{r|l} \textbf{NI} & \\ & \\ & n[0] \end{array} \qquad \begin{array}{r|l} \textbf{Ca} & /n/[s] \\ & \\ & \backslash n\backslash[s] \end{array} \qquad \begin{array}{r|l} \textbf{Cb} & /n/[a] \\ & /\!\!/P/\!\!/_a \\ & \\ & \backslash\!\!\backslash P\backslash\!\!\backslash_a \end{array}$$

For the star-rules, $/\!\!/P/\!\!/_t$ and $/\!\!/Q/\!\!/_t$ may be either P_t and Q_t, or $\overline{P}_t$ and $\overline{Q}_t$. Then,

$N\upsilon K_4$ adds →I4 and →E4 with K

$N\upsilon K_*$ adds →I∗ and →E∗ with K

$N\upsilon N_*$ adds →I∗ and →E∗ with NI

$N\upsilon N_4$ adds →I4 and →E4 with NI and Ca

A system with CL would add both Ca and Cb. As a simplification, in the first cases, one might eliminate rule K, and delete the normality requirement from other rules. In these systems, every subscript is 0, appears in a premise, or appears in the t-place of an accessible assumption for →I. Where the members of Γ and A are without overlines or subscripts, let Γ_0 be the members of Γ, each with subscript 0. Then,

NυX $\Gamma \vdash_{N\upsilon X} A$ iff there is an $N\upsilon X$ derivation of A_0 from Γ_0.

Derived rules are as one would expect. Two-way derived rules carry over from *CL* with overlines and subscripts constant throughout. Thus, e.g.,

$$\begin{array}{ll} \textbf{Impl} & /P \supset Q/_s \;\lhd\rhd\; /\neg P \vee Q/_s \\ & /\neg P \supset Q/_s \;\lhd\rhd\; /P \vee Q/_s \end{array}$$

MT, NB and DS appear in the forms,

$$\begin{array}{r|l} \textbf{MT} & /P \supset Q/_s \\ & \backslash\neg Q\backslash_s \\ & \\ & /\neg P/_s \end{array} \qquad \begin{array}{r|l} \textbf{NB} & /P \equiv Q/_s \\ & \backslash\neg P\backslash_s \\ & \\ & /\neg Q/_s \end{array} \quad \begin{array}{|l} /P \equiv Q/_s \\ \backslash\neg Q\backslash_s \\ \\ /\neg P/_s \end{array} \qquad \begin{array}{r|l} \textbf{DS} & /P \vee Q/_s \\ & \backslash\neg P\backslash_s \\ & \\ & /Q/_s \end{array} \quad \begin{array}{|l} /P \vee Q/_s \\ \backslash\neg Q\backslash_s \\ \\ /P/_s \end{array}$$

System NIx. These systems take over rules for ¬, ∨ and ∧ from before, and then add from the following in the natural way.

$$\begin{array}{r|l} \sqsupset\textbf{I} & \begin{array}{|l} s.t \\ P_t \\ \hline \\ Q_t \end{array} \\ & (P \sqsupset Q)_s \end{array} \qquad\qquad \begin{array}{r|l} \sqsupset\textbf{E} & (P \sqsupset Q)_s \\ & s.t \\ & P_t \\ & \\ & Q_t \end{array}$$

where t does not appear in any undischarged premise or assumption

$$\begin{array}{l|l} \textbf{AM}\rho & \\ & s.s \end{array} \qquad \begin{array}{l|l} \textbf{AM}\tau & s.t \\ & t.u \\ & s.u \end{array} \qquad \begin{array}{l|l|l} \textbf{H}_{\mathrm{I}} & P_s & \overline{P}_t \\ & s.t & s.t \\ & P_t & \overline{P}_s \end{array} \qquad \begin{array}{l|l} \textbf{D} & P \\ & \overline{P} \end{array}$$

$$\begin{array}{l|l} \overline{\sqsupset \mathrm{I}} & \quad | \; P_s \\ & \quad | \; \overline{Q}_s \\ & (\overline{P \sqsupset Q})_s \end{array} \qquad \begin{array}{l|l} \overline{\sqsupset \mathrm{E}} & (\overline{P \sqsupset Q})_s \\ & P_s \\ & \overline{Q}_s \end{array} \qquad \begin{array}{l|l} \overline{\sqsupset \mathrm{I}}_W & s.t \\ & P_t \\ & \overline{Q}_t \\ & (\overline{P \sqsupset Q})_s \end{array} \qquad \begin{array}{l|l} \overline{\sqsupset \mathrm{E}}_W & (\overline{P \sqsupset Q})_s \\ & \quad | \; s.t \\ & \quad | \; P_t \\ & \quad | \; \overline{Q}_t \\ & \quad | \; A_u \\ & A_u \end{array}$$

where t does not appear in any undischarged premise or assumption and is not u

Each of the NIx systems have ⊐I, ⊐E, AMρ, AMτ and H$_I$. NI_4 then takes $\overline{\sqsupset \mathrm{I}}$ and $\overline{\sqsupset \mathrm{E}}$, NI_3 adds D. NI_W substitutes $\overline{\sqsupset \mathrm{I}}_W$ and $\overline{\sqsupset \mathrm{E}}_W$ in the four-valued system. Validity is as before.

Examples. Here are a few cases where the logics do not all have the same results.

$P \rightarrow Q \vdash_{NvX_*} \neg Q \rightarrow \neg P$

1	$(P \rightarrow Q)_0$	P
2	$n[0]$	NI or K
3	$\quad \neg Q_1$	A (g, $\rightarrow$I$*$)
4	$\qquad \overline{P}_1$	A (c, $\neg$I)
5	$\qquad \overline{Q}_1$	2,1,4 $\rightarrow$E$*$
6	$\qquad \neg Q_1$	3 R
7	$\quad \neg P_1$	4-6 $\neg$I
8	$(\neg Q \rightarrow \neg P)_0$	2,3-7 $\rightarrow$I$*$

This derivation works with either (K) or (NI), but does not go through in the 4-systems insofar as there is no "purchase" for application of $\rightarrow$E4 with (1) and only $\overline{P}_1$, rather than P_1, at (4).

$P \wedge \neg Q \vdash_{NvX_4} \neg(P \to Q)$

1	$(P \wedge \neg Q)_0$	P
2	$n[0]$	NI
3	$\overline{n}[0]$	Ca or directly by K
4	$\vert\ \overline{(P \to Q)}_0$	A (c, $\neg$I)
5	$\vert\ P_0$	1 $\wedge$E
6	$\vert\ \overline{Q}_0$	3,4,5 $\to$E4
7	$\vert\ \neg Q_0$	1 $\wedge$E
8	$\neg(P \to Q)_0$	4-7 $\neg$I

This derivation works with either (NI) and (Ca) or (K). It is blocked in either star system insofar as the contradiction does not arise: by $\to$E$*$, we might get Q_0 at (4), but this does not contradict $\neg Q_0$ for $\neg$I.

$\vdash_{NvK_x} [(P \to Q) \wedge (Q \to R)] \to (P \to R)$

1	$n[0]$	K
2	$\vert\ [(P \to Q) \wedge (Q \to R)]_1$	A (g, $\to$Ix)
3	$\vert\ n[1]$	K
4	$\vert\ \vert\ P_2$	A (g, $\to$Ix)
5	$\vert\ \vert\ (P \to Q)_1$	2 $\wedge$E
6	$\vert\ \vert\ Q_2$	3,4,5 $\to$Ex
7	$\vert\ \vert\ (Q \to R)_1$	2 $\wedge$E
8	$\vert\ \vert\ R_2$	3,6,7 $\to$Ex
9	$\vert\ (P \to R)_1$	3,4-8 $\to$Ix
10	$([(P \to Q) \wedge (Q \to R)] \to (P \to R))_0$	1,2-9 $\to$Ix

This derivation works with either the star- or 4-rules. But it works only with (K) insofar as $s = 1$ for lines (6), (8) and (9). And, finally, a couple cases to show $\neg(A \sqsupset B)_s \;\triangleleft\triangleright\; (A \sqsupset \neg B)_s$ in NI_W

1	$\neg(A \sqsupset B)_s$	P
2	$\vert\ s.t$	A (g, $\sqsupset$I)
3	$\vert\ A_t$	
4	$\vert\ \vert\ \overline{B}_t$	A (c, $\neg$I)
5	$\vert\ \vert\ \overline{(A \sqsupset B)}_s$	2,3,4 $\overline{\sqsupset}$I
6	$\vert\ \vert\ \neg(A \sqsupset B)_s$	1 R
7	$\vert\ \neg B_t$	4-6 $\neg$I
8	$(A \sqsupset \neg B)_s$	2-7 $\sqsupset$I

1	$(A \sqsupset \neg B)_s$	P
2	$(\overline{A \sqsupset B})_s$	A (c, $\neg$I)
3	$s.t$	A g, 2 $\overline{\sqsupset \text{E}}$)
4	A_t	
5	$\overline{B}_t$	
6	$(\overline{A \sqsupset B})_s$	A $\neg$I
7	$\neg B_t$	1,3,4 $\sqsupset$E
8	$\overline{B}_t$	5 R
9	$\neg(A \sqsupset B)_s$	6-8 $\neg$I
10	$\neg(A \sqsupset B)_s$	2,3-9 $\overline{\sqsupset \text{E}}$
11	$(\overline{A \sqsupset B})_s$	2 R
12	$\neg(A \sqsupset B)_s$	2-11 $\neg$I

8 Mainstream Relevant Logics: *Bx* (ch. 10,11)

The treatment here for Priest's chapter 11 is minimal: there are only resources for *CK* with applications in chapter 11, as well as chapter 10. I follow Priest in developing the star-semantics on its own terms, and pick up the four-valued semantics again in the next section.

8.1 Language / Semantic Notions

LBx The VOCABULARY consists of propositional parameters $p_0, p_1 \ldots$ with the operators, $\neg$, $\wedge$, $\vee$, $\rightarrow$, (and $>$). Each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A > B)$. $A \supset B$ abbreviates $\neg A \vee B$, and $A \equiv B$ abbreviates $(A \supset B) \wedge (B \supset A)$.

IBrx Without '$>$' in the language, an INTERPRETATION is $\langle W, N, R, *, \sqsubseteq, v\rangle$ where W is a set of worlds; N is a subset of W; R is a subset of $W^3 = W \times W \times W$; $*$ is a function from worlds to worlds such that $w^{**} = w$; and v is a function such that for any $w \in W$ and p, $v_w(p) = 1$ or $v_w(p) = 0$. $\sqsubseteq$ is a reflexive and transitive relation on W such that if $a \sqsubseteq b$ then $a \trianglelefteq b$ and $b^* \trianglelefteq a^*$, where,

$$a \trianglelefteq b = \begin{cases} \text{if } v_a(p) = 1 \text{ then } v_b(p) = 1 \\ \text{if } bRxy \text{ and } a \notin N\text{, then } aRxy \\ \text{if } bRxy \text{ and } a \in N \text{ then } x \sqsubseteq y \end{cases}$$

As a constraint on interpretations, we require also, NC For any $w \in N$, $Rwxy$ iff $x = y$

Where x is empty or indicates some combination of the following constraints,

(C8) If $Rabc$, then Rac^*b^*

(C9) If there is an x such that $Rabx$ and $Rxcd$ then there is a y such that $Racy$ and $Rbyd$

(C10) If there is an x such that $Rabx$ and $Rxcd$ then there is a y such that $Rbcy$ and $Rayd$

(C11) If $Rabc$ then there is an x such that $Rabx$ and $Rxbc$

(C12) If $Rabc$ then there is an $x \sqsupseteq a$ such that $Rbxc$

(C13) If $x \in N$, $x^* \sqsubseteq x$.

(C14) For any x, if $x \in N$, $x^* \sqsubseteq x$, and if $x \notin N$, xRx^*x.

(C15) If $Rabc$ then $a \sqsubseteq c$.

(C16) If $Rabc$ then $a \sqsubseteq c$ or $b \sqsubseteq c$.

$\langle W, N, R, *, \sqsubseteq, v\rangle$ is a Bx interpretation when it meets the constraints from x. System B has none of the extra constraints; other systems add from the extra constraints as described in Priest. In particular, B_R is B_{C8-C12}.

IBcx When '>' is in the language, an interpretation is $\langle W, N, R, \{R_A \mid A \in \Im\}, *, v\rangle$, where $\Im$ is the set of all formulas and R_A is a subset of W^2. Condition NC remains in place, but none of C8 - C16. That is all for B_C (what Priest calls C_B). Where $f_A(w) = \{x \in W \mid wR_Ax\}$, and $[A] = \{x \in W \mid v_w(A) = 1\}$, B_{C^+} adds the constraints,

(1) For any $w \in N$, $f_A(w) \subseteq [A]$

(2) For any $w \in N$, if $w \in [A]$, then $w \in f_A(w)$

TB For complex expressions,

($\neg$) $v_w(\neg A) = 1$ if $v_{w^*}(A) = 0$, and 0 otherwise.

($\wedge$) $v_w(A \wedge B) = 1$ if $v_w(A) = 1$ and $v_w(B) = 1$, and 0 otherwise.

($\vee$) $v_w(A \vee B) = 1$ if $v_w(A) = 1$ or $v_w(B) = 1$, and 0 otherwise.

($\rightarrow$) $v_w(A \rightarrow B) = 1$ iff there are no $x, y \in W$ such that $Rwxy$ and $v_x(A) = 1$ but $v_y(B) = 0$.

($>$) $v_w(A > B) = 1$ iff there is no $x \in W$ such that wR_Ax and $v_x(B) = 0$.

For a set Γ of formulas, $v_w(\Gamma) = 1$ iff $v_w(A) = 1$ for each $A \in \Gamma$; then,

VBx $\Gamma \models_{Bx} A$ iff there is no Bx interpretation $\langle W, N, R, *, \sqsubseteq, v\rangle$ / $\langle W, N, R, \{R_A \mid A \in \Im\}, *, v\rangle$ and $w \in N$ such that $v_w(\Gamma) = 1$ and $v_w(A) = 0$.

8.2 Natural Derivations: *NBx*

Allow subscripts of the sort i and $i^{\#}$. Where s is a subscript i or $i^{\#}$, $\overline{s}$ is the other. Say s is "introduced" as a subscript when either s or $\overline{s}$ is a subscript. For subscripts s, t, u allow also expressions of the sort $s \simeq t$, $s.t.u$ and $A_{s/t}$. Let $\mathcal{P}(s)$ be any expression in which s appears, and $\mathcal{P}(t)$ the same expression with one or more instances of s replaced by t.

$$\textbf{R}\ \begin{array}{|l} P_s \\ \\ P_s \end{array} \qquad \neg\textbf{I}\ \begin{array}{|l} \begin{array}{|l} P_{\overline{s}} \\ \hline \\ Q_t \\ \neg Q_{\overline{t}} \end{array} \\ \neg P_s \end{array} \qquad \neg\textbf{E}\ \begin{array}{|l} \begin{array}{|l} \neg P_{\overline{s}} \\ \hline \\ Q_t \\ \neg Q_{\overline{t}} \end{array} \\ P_s \end{array}$$

$$\wedge\textbf{I}\ \begin{array}{|l} P_s \\ Q_s \\ \\ (P \wedge Q)_s \end{array} \qquad \wedge\textbf{E}\ \begin{array}{|l} (P \wedge Q)_s \\ \\ P_s \end{array} \qquad \wedge\textbf{E}\ \begin{array}{|l} (P \wedge Q)_s \\ \\ Q_s \end{array}$$

$$\vee\textbf{I}\ \begin{array}{|l} P_s \\ \\ (P \vee Q)_s \end{array} \qquad \vee\textbf{I}\ \begin{array}{|l} P_s \\ \\ (Q \vee P)_s \end{array} \qquad \vee\textbf{E}\ \begin{array}{|l} (P \vee Q)_s \\ \begin{array}{|l} P_s \\ \hline \\ R_t \end{array} \\ \begin{array}{|l} Q_s \\ \hline \\ R_t \end{array} \\ R_t \end{array}$$

$$\supset\textbf{I}\ \begin{array}{|l} \begin{array}{|l} P_{\overline{s}} \\ \hline \\ Q_s \end{array} \\ (P \supset Q)_s \end{array} \qquad \supset\textbf{E}\ \begin{array}{|l} (P \supset Q)_s \\ P_{\overline{s}} \\ \\ Q_s \end{array}$$

$$\equiv\textbf{I}\ \begin{array}{|l} \begin{array}{|l} P_{\overline{s}} \\ \hline \\ Q_s \end{array} \\ \begin{array}{|l} Q_{\overline{s}} \\ \hline \\ P_s \end{array} \\ (P \equiv Q)_s \end{array} \qquad \equiv\textbf{E}\ \begin{array}{|l} (P \equiv Q)_s \\ P_{\overline{s}} \\ \\ Q_s \end{array} \qquad \equiv\textbf{E}\ \begin{array}{|l} (P \equiv Q)_s \\ Q_{\overline{s}} \\ \\ P_s \end{array}$$

→I
$$\begin{array}{l|l} & \begin{array}{|l} s.t.u \\ P_t \\ \hline \\ Q_u \end{array} \\ & (P \rightarrow Q)_s \end{array}$$
where t and u are not introduced in any undischarged premise or assumption

→E
$$\begin{array}{|l} s.t.u \\ (P \rightarrow Q)_s \\ P_t \\ \\ Q_u \end{array}$$

$\not\rightarrow$I
$$\begin{array}{|l} \overline{s}.t.u \\ P_t \\ \neg Q_{\overline{u}} \\ \\ \neg(P \rightarrow Q)_s \end{array}$$

$\not\rightarrow$E
$$\begin{array}{|l} \neg(P \rightarrow Q)_s \\ \begin{array}{|l} \overline{s}.t.u \\ P_t \\ \neg Q_{\overline{u}} \\ \hline \\ R_v \end{array} \\ R_v \end{array}$$
where t and u are not introduced in any undischarged premise or assumption or by v

0I
$$\begin{array}{|l} s \simeq t \\ \\ 0.s.t \end{array}$$

0E
$$\begin{array}{|l} 0.s.t \\ \\ s \simeq t \end{array}$$

$\simeq$I
$$\begin{array}{|l} \\ \\ s \simeq s \end{array}$$

$\simeq$E
$$\begin{array}{|l} s \simeq t \\ \mathcal{P}(s) \\ \\ \mathcal{P}(t) \end{array} \qquad \begin{array}{|l} s \simeq t \\ \mathcal{P}(\overline{s}) \\ \\ \mathcal{P}(\overline{t}) \end{array}$$

These are the rules of *NB*, where ⊃I, ⊃E, ≡I, ≡E and, as we shall see, $\not\rightarrow$I and $\not\rightarrow$E are derived. With $s \simeq t$, we can introduce $s \simeq s$ by $\simeq$I, and then get $t \simeq s$ by $\simeq$E; so informally, we let $\simeq$E include also a derived rule that reverses order around '$\simeq$' – using $s \simeq t$ to replace some instance(s) of t ($\overline{t}$) with s ($\overline{s}$). As usual, subscripts are 0 or introduced in an assumption that requires new subscripts (and similarly for the following). To make things easier to follow, cite lines for →E only in the order listed above: first access, then the conditional, then the antecedent.

For relevant systems NB_x, allow expressions of the sort, $s \preceq t$ and $s \not\simeq t$. The latter contradicts $s \simeq t$ in ¬I and ¬E.[5] Then include rules from the following as appropriate.

AM9
$$\begin{array}{|l} s.t.x \\ x.u.v \\ \begin{array}{|l} s.u.y \\ t.y.v \\ \hline \\ P_w \end{array} \\ P_w \end{array}$$

AM10
$$\begin{array}{|l} s.t.x \\ x.u.v \\ \begin{array}{|l} t.u.y \\ s.y.v \\ \hline \\ P_w \end{array} \\ P_w \end{array}$$

AM11
$$\begin{array}{|l} s.t.u \\ \begin{array}{|l} s.t.y \\ y.t.u \\ \hline \\ P_w \end{array} \\ P_w \end{array}$$

AM12
$$\begin{array}{|l} s.t.u \\ \begin{array}{|l} s \preceq y \\ t.y.u \\ \hline \\ P_w \end{array} \\ P_w \end{array}$$

[5]We might allow a generic subscript z such that any $s \simeq t$ is $(s \simeq t)_z$ and $s \not\simeq t$ is $\neg(s \simeq t)_{z\#}$. Then the negation rules apply as stated.

AM13
$$\begin{array}{|l} \\ 0^{\#} \preceq 0 \end{array}$$

AM14
$$\begin{array}{|l} \begin{array}{|l} s \simeq 0 \\ \underline{\bar{s} \preceq s} \\ P_w \end{array} \\ \begin{array}{|l} s \not\simeq 0 \\ \underline{s.\bar{s}.s} \\ P_w \end{array} \\ P_w \end{array}$$

AM15
$$\begin{array}{|l} s.t.u \\ \\ s \preceq u \end{array}$$

AM16
$$\begin{array}{|l} s.t.u \\ \begin{array}{|l} \underline{s \preceq u} \\ P_w \end{array} \\ \begin{array}{|l} \underline{t \preceq u} \\ P_w \end{array} \\ P_w \end{array}$$

AM8
$$\begin{array}{|l} s.t.u \\ \\ s.\bar{u}.\bar{t} \end{array}$$

$\preceq$**E**
$$\begin{array}{|l} s \preceq t \\ P_s \\ \\ P_t \end{array}$$

$\preceq^{\#}$
$$\begin{array}{|l} s \preceq t \\ \\ \bar{t} \preceq \bar{s} \end{array}$$

$\preceq^{R}$
$$\begin{array}{|l} s \not\simeq 0 \\ s \preceq t \\ t.u.v \\ \\ s.u.v \end{array} \qquad \begin{array}{|l} s \simeq 0 \\ s \preceq t \\ t.u.v \\ \\ u \preceq v \end{array}$$

For AM9, AM10, AM11 and AM12, y is not introduced in any undischarged premise or assumption, or by w. Rules for $\preceq$ are always included with any of AM12 - AM16.[6]

Conditional systems. For the systems NB_{C_x} revert to the rules of *NB*. Then add >I and >E. As we show just below, $\not>$I and $\not>$E are derived.

>I
$$\begin{array}{|l} \begin{array}{|l} \underline{P_{s/t}} \\ \\ Q_t \end{array} \\ (P > Q)_s \end{array}$$
where t is not introduced in any undischarged premise or assumption

>E
$$\begin{array}{|l} (P > Q)_s \\ P_{s/t} \\ \\ Q_t \end{array}$$

$\not>$**I**
$$\begin{array}{|l} P_{\bar{s}/t} \\ \neg Q_{\bar{t}} \\ \\ \neg(P > Q)_s \end{array}$$

$\not>$**E**
$$\begin{array}{|l} \neg(P > Q)_s \\ \begin{array}{|l} P_{\bar{s}/t} \\ \underline{\neg Q_{\bar{t}}} \\ \\ R_u \end{array} \\ R_u \end{array}$$
where t is not introduced in any undischarged premise or assumption, or by u

As before, corresponding to constraints (1) and (2) for the C^+ system, are AMP1 and AMP2, now restricted to apply just at the normal world 0.

AMP1
$$\begin{array}{|l} P_{0/t} \\ \\ P_t \end{array}$$

AMP2
$$\begin{array}{|l} P_0 \\ \\ P_{0/0} \end{array}$$

[6] There are also rules, $\preceq^{\rho}$ according to which $\vdash s \preceq s$ and $\preceq^{\tau}$ according to which $s \preceq t$, $t \preceq u \vdash s \preceq u$. But these do not normally play a role in derivations.

Where Γ is a set of unsubscripted formulas, let Γ_0 be those same formulas, each with subscript 0. Then,

NBx $\Gamma \vdash_{NBx} A$ iff there is an NBx derivation of A_0 from the members of Γ_0.

Derived rules carry over much as one would expect. Thus, e.g.,

$$\textbf{MT}\ \begin{array}{|l} (P \supset Q)_s \\ \neg Q_{\overline{s}} \\ \\ \neg P_s \end{array} \qquad \textbf{NB}\ \begin{array}{|l} (P \equiv Q)_s \\ \neg P_{\overline{s}} \\ \\ \neg Q_s \end{array} \quad \begin{array}{|l} (P \equiv Q)_s \\ \neg Q_{\overline{s}} \\ \\ \neg P_s \end{array} \qquad \textbf{DS}\ \begin{array}{|l} (P \vee Q)_s \\ \neg P_{\overline{s}} \\ \\ Q_s \end{array} \quad \begin{array}{|l} (P \vee Q)_s \\ \neg Q_{\overline{s}} \\ \\ P_s \end{array}$$

Impl $\quad (P \supset Q)_s \;\lhd\rhd\; (\neg P \vee Q)_s$
$\quad (\neg P \supset Q)_s \;\lhd\rhd\; (P \vee Q)_s$

Examples. First, $\not\to$I, $\not\to$E, $\not>$I and $\not>$E are derived rules in NB_x and NB_{Cx}.

$\not\to$**I**

1	$\overline{s}.t.u$	P
2	P_t	P
3	$\neg Q_{\overline{u}}$	P
4	$\mid\ (P \to Q)_{\overline{s}}$	A (c, $\neg$I)
5	$\mid\ Q_u$	1,4,2 $\to$E
6	$\mid\ \neg Q_{\overline{u}}$	3 R
7	$\neg(P \to Q)_s$	4-6 $\neg$I

$\not\to$E

1	$\neg(P \to Q)_s$	P
2	$\mid\ \neg R_{\overline{v}}$	A (c, $\neg$E)
3	$\mid\ \mid\ \overline{s}.t.u$	A (g, $\to$I)
4	$\mid\ \mid\ P_t$	
5	$\mid\ \mid\ \mid\ \neg Q_{\overline{u}}$	A (c, $\neg$E)
	$\mid\ \mid\ \mid\ \vdots$	with 1,3,4,5
6	$\mid\ \mid\ \mid\ R_v$	as for $\not\to$E
7	$\mid\ \mid\ \mid\ \neg R_{\overline{v}}$	2 R
8	$\mid\ \mid\ Q_u$	5-7 $\neg$E
9	$\mid\ (P \to Q)_{\overline{s}}$	3-8 $\to$I
10	$\mid\ \neg(P \to Q)_s$	1 R
11	R_v	2-10 $\neg$E

$\not>$I

1	$P_{\overline{s}/t}$	P
2	$\neg Q_{\overline{t}}$	P
3	$\quad (P > Q)_{\overline{s}}$	A (c, ¬I)
4	$\quad Q_t$	1,3 >E
5	$\quad \neg Q_{\overline{t}}$	2 R
6	$\neg(P > Q)_s$	3-5 ¬I

$\not>$E

1	$\neg(P > Q)_s$	P
2	$\quad \neg R_{\overline{u}}$	A (c, ¬E)
3	$\quad\quad P_{\overline{s}/t}$	A (g, >I)
4	$\quad\quad\quad \neg Q_{\overline{t}}$	A (c, ¬E)
	$\quad\quad\quad \vdots$	with 1,3,4
5	$\quad\quad\quad R_u$	as for $\not>$E
6	$\quad\quad\quad \neg R_{\overline{u}}$	2 R
7	$\quad\quad Q_t$	4-6 ¬E
8	$\quad (P > Q)_{\overline{s}}$	3-7 >I
9	$\quad \neg(P > Q)_s$	1 R
10	R_u	2-9 ¬E

Note the way overlines work (much the way slashes worked before). For $\not\to$E, note that the application of →I depends on the restriction that t and u are not introduced by v; and similarly, for $\not>$E the application of >I depends on the restriction that t is not introduced by u.

As further examples, here are a few key results that parallel ones from Priest's text.

A3 $\vdash_{NBx} (A \wedge B) \to A$

1	$\quad 0.1.2$	A (g, →I)
2	$\quad (A \wedge B)_1$	
3	$\quad A_1$	2 ∧E
4	$\quad 1 \simeq 2$	1 0E
5	$\quad A_2$	3,4 ≃E
6	$[(A \wedge B) \to A]_0$	1-5 →I

A5 $\vdash_{NBx} [(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

1	$\quad$ 0.1.2	A (g, $\to$I)
2	$\quad [(A \to B) \wedge (A \to C)]_1$	
3	$\qquad$ 2.3.4	A (g, $\to$I)
4	$\qquad A_3$	
5	$\qquad 1 \simeq 2$	1 0E
6	$\qquad$ 1.3.4	3,5 $\simeq$E
7	$\qquad (A \to B)_1$	2 $\wedge$E
8	$\qquad (A \to C)_1$	2 $\wedge$E
9	$\qquad B_4$	6,7,4 $\to$E
10	$\qquad C_4$	6,8,4 $\to$E
11	$\qquad (B \wedge C)_4$	9,10 $\wedge$I
12	$\quad [A \to (B \wedge C)]_2$	3-11 $\to$I
13	$([(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)])_0$	1-12 $\to$I

R5 $(A \to \neg B) \vdash_{NBx} (B \to \neg A)$

1	$(A \to \neg B)_0$	P
2	$\quad$ 0.1.2	A (g, $\to$I)
3	$\quad B_1$	
4	$\qquad A_{2^\#}$	A (c, $\neg$I)
5	$\qquad 2^\# \simeq 2^\#$	$\simeq$I
6	$\qquad 0.2^\#.2^\#$	5 0I
7	$\qquad \neg B_{2^\#}$	6,1,4 $\to$E
8	$\qquad 1 \simeq 2$	2 0E
9	$\qquad B_2$	3,8 $\simeq$E
10	$\quad \neg A_2$	4-9 $\neg$I
11	$(B \to \neg A)_0$	2-10 $\to$I

A9 $\vdash_{NB_9} (A \to B) \to [(B \to C) \to (A \to C)]$

1	$0.1.2$	A (g, $\to$I)
2	$(A \to B)_1$	
3	$2.3.4$	A (g, $\to$I)
4	$(B \to C)_3$	
5	$4.5.6$	A (g, $\to$I)
6	A_5	
7	$1 \simeq 2$	1 0E
8	$(A \to B)_2$	2,7 $\simeq$E
9	$2.5.7$	A (g, 3,5 AM9)
10	$3.7.6$	
11	B_7	9,8,6 $\to$E
12	C_6	10,4,11 $\to$E
13	C_6	3,5,9-12 AM9
14	$(A \to C)_4$	5-13 $\to$I
15	$[(B \to C) \to (A \to C)]_2$	3-14 $\to$I
16	$((A \to B) \to [(B \to C) \to (A \to C)])_0$	1-15 $\to$I

$\vdash_{NB_R} (\neg A \to A) \to A$

1	$0.1.2$	A (g, $\to$I)
2	$(\neg A \to A)_1$	
3	$0.2^{\#}.1^{\#}$	1 AM8
4	$0.2^{\#}.3$	A (g, 3 AM11)
5	$3.2^{\#}.1^{\#}$	
6	$3.1.2$	5 AM8
7	$3 \preceq 4$	A (g, 6 AM12)
8	$1.4.2$	
9	$\neg A_{2^{\#}}$	A (c, $\neg$E)
10	$2^{\#} \simeq 3$	4 0E
11	$\neg A_3$	9,10 $\simeq$E
12	$\neg A_4$	7,11 $\preceq$E
13	A_2	8,2,12 $\to$E
14	$\neg A_{2^{\#}}$	9R
15	A_2	9-14 $\neg$E
16	A_2	6,7-15 AM12
17	A_2	3,4-16 AM11
18	$[(\neg A \to A) \to A]_0$	1-17 $\to$I

A14 $\vdash_{NB_{14}} (A \to \neg A) \to \neg A$

1	$0.1.2$	A (g, $\to$I)
2	$(A \to \neg A)_1$	
3	$1 \simeq 2$	1 0E
4	$2 \simeq 0$	A (g, AM14)
5	$2^{\#} \preceq 2$	
6	$A_{2^{\#}}$	A (c, $\neg$I)
7	$1 \simeq 0$	3,4 $\simeq$E
8	$(A \to \neg A)_0$	2,7 $\simeq$E
9	A_2	5,6 $\preceq$E
10	A_1	9,3 $\simeq$E
11	$\neg A_2$	1,8,10 $\to$E
12	$A_{2^{\#}}$	6R
13	$\neg A_2$	6-12 $\neg$I
14	$2 \not\simeq 0$	A (g, AM14)
15	$2.2^{\#}.2$	
16	$A_{2^{\#}}$	A (c, $\neg$I)
17	$(A \to \neg A)_2$	2,3 $\simeq$E
18	$\neg A_2$	15,17,16 $\to$E
19	$A_{2^{\#}}$	16 R
20	$\neg A_2$	16-19 $\neg$I
21	$\neg A_2$	4-13,14-20 AM14
22	$[(A \to \neg A) \to \neg A]_0$	1-21 $\to$I

9 Four-Valued Relevant Logics: ***R4x*** (ch. 10,11)

Though Priest does not do so — and it has been suggested that it cannot reasonably be done [4], relevant systems are capable of a four-valued treatment. Thus, to make contact with what has gone before, and contact with some of my own suggestions for the significance of relevant semantics [5], a four-valued approach is developed. The discussion is restricted to (standard) logics in the range DW - R — though it might be extended beyond.

9.1 Language / Semantic Notions

LR4 The VOCABULARY consists of propositional parameters $p_0, p_1 \ldots$ with the operators $\neg$, $\wedge$, $\vee$, and $\to$. Each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, and $(A \to B)$. $A \supset B$ abbreviates $\neg A \vee B$. In the extended case, the language includes $\Box$; then if A is a formula, so is $\Box A$; and $\Diamond A$ abbreviates $\neg\Box\neg A$. If A is a formula so formed, so is $\overline{A}$.

Let $/A/$ and $\backslash A\backslash$ represent either A or $\overline{A}$ where what is represented is constant in a given context, but $/A/$ and $\backslash A\backslash$ are opposite. And similarly for other expressions with overlines as below.

IR4 Without $\Box$ in the language, an INTERPRETATION is $\langle W, N, \overline{N}, R, \overline{R}, \preceq, v\rangle$ where W is a set of worlds; $N, \overline{N} \subseteq W$ are normal worlds for truth and non-falsity respectively; $R, \overline{R} \subseteq W^3$ are access relations for truth and non-falsity respectively; and v is a valuation which assigns to each parameter some subset of $\{0, 1\}$ at each $w \in W$. $\preceq$ encompasses the inclusion relations $\leq$, $\leq^*$ and $\leq^\sharp$, constrained so that (with the conversion between h and v from below),

$$a \leq b \Rightarrow \begin{cases} \text{if } h_a(p) = 1 \text{ then } h_b(p) = 1 \text{ and if } h_b(\overline{p}) = 1 \text{ then } h_a(\overline{p}) = 1 \\ \text{if } bRxy \text{ then } aRxy \text{ if } a \notin N\text{, otherwise if } bRxy \text{ then } x \leq y \\ \text{if } a\overline{R}xy \text{ then } b\overline{R}xy \text{ if } b \notin \overline{N}\text{, otherwise if } a\overline{R}xy \text{ then } x \leq y \end{cases}$$

$$a \leq^* b \Rightarrow \begin{cases} \text{if } h_a(p) = 1 \text{ then } h_b(\overline{p}) = 1 \text{ and if } h_b(p) = 1 \text{ then } h_a(\overline{p}) = 1 \\ \text{if } b\overline{R}xy \text{ then } aRxy \text{ if } a \notin N\text{, otherwise if } b\overline{R}xy \text{ then } x \leq y \\ \text{if } a\overline{R}xy \text{ then } bRxy \text{ if } b \notin N\text{, otherwise if } a\overline{R}xy \text{ then } x \leq y \end{cases}$$

$$a \leq^\sharp b \Rightarrow \begin{cases} \text{if } h_a(\overline{p}) = 1 \text{ then } h_b(p) = 1 \text{ and if } h_b(\overline{p}) = 1 \text{ then } h_a(p) = 1 \\ \text{if } bRxy \text{ then } a\overline{R}xy \text{ if } a \notin \overline{N}\text{, otherwise if } bRxy \text{ then } x \leq y \\ \text{if } aRxy \text{ then } b\overline{R}xy \text{ if } b \notin \overline{N}\text{, otherwise if } aRxy \text{ then } x \leq y \end{cases}$$

As additional constraints on interpretations, we may require any of,

NC For any $w \in /N/$, $w/R/xy$ iff $x = y$

C^9_{10} If $a/R/bx$ and $xRcd$ then there is a y such that $bRcy$ and $a/R/yd$, and a z such that $bRzd$ and $a/R/cz$. And if $a/R/xb$ and $x\overline{R}cd$ then there is a y such that $b\overline{R}cy$ and $a/R/yd$, and a z such that $b\overline{R}zd$ and $a/R/cz$.

C11 If $a/R/bc$ then there is a y such that $a/R/by$ and $yRbc$ and a z such that $a/R/zc$ and $z\overline{R}bc$.

C12 If $aRbc$ then for some $y \geq a$, $bRyc$, and for some $z \geq^* a$, $c\overline{R}bz$. And if $a\overline{R}bc$ then for some $y \geq^\sharp a$, $bRyc$, and for some $z \leq a$, $c\overline{R}bz$

CL (i) $w \in N$ iff $w \in \overline{N}$

(ii) for any $w \in N$, $0 \notin v_w(p)$ iff $1 \in v_w(p)$

In this case, the base standard system is DW and includes just NC. Other regular relevant systems add from C9 - C12 in the usual way [8]. The FA systems from [5] drop NC but may include C9 - C12; the FB systems include NC, and might include any of the other constraints, including CL.

Where the language includes $\Box$, FB interpretations may be extended to be of the sort, $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, v\rangle$ where $M \subseteq W$ is a modal access relation. Interpretations are subject to,

MC If $w \in /N/$ and wMx, then $x \in /N/$

and optionally standard modal constraints of the sort,

- ρ Reflexivity: for all x, xMx.
- σ Symmetry: for all x, y if xMy then yMx.
- τ Transitivity: for all x, y, z if xMy and yMz then xMz.

HR4 For complex expressions,

- (B) $h_w(p) = 1$ iff $1 \in v_w(p)$; $h_w(\overline{p}) = 1$ iff $0 \notin v_w(p)$
- ($\neg$) $h_w(/\neg P/) = 1$ iff $h_w(\backslash P\backslash) = 0$
- ($\wedge$) $h_w(/P \wedge Q/) = 1$ iff $h_w(/P/) = 1$ and $h_w(/Q/) = 1$
- ($\vee$) $h_w(/P \vee Q/) = 1$ iff $h_w(/P/) = 1$ or $h_w(/Q/) = 1$
- ($\rightarrow$) $h_w(/P \rightarrow Q/) = 1$ iff there are no $x, y \in W$ such that $w/R/xy$ and $h_x(P) = 1$ but $h_y(Q) = 0$, or $h_y(\overline{P}) = 1$ but $h_x(\overline{Q}) = 0$
- ($\Box$) $h_w(/\Box P/) = 1$ iff there is no $x \in W$ such that wMx and $h_x(/P/) = 0$

For a set Γ of formulas, $h_w(\Gamma) = 1$ iff $h_w(/P/) = 1$ for each $/P/ \in \Gamma$; then,

VR4 $\Gamma \models_{NR4x} P$ iff there is no R4x interpretation $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle$ and $w \in N$ such that $h_w(\Gamma) = 1$ but $h_w(P) = 0$.

9.2 Natural Derivations: *NR4x*

Allow subscripts and expressions of the sort $s.t$, $/s.t.u/$, $s \geq t$, $s \geq^* t$, and $s \geq^\sharp t$. Allow also $/n/[s]$ and $\sim/n/[s]$; to say that a world is or is not in $/N/$; these contradict in $\neg$I and $\neg$E.[7]

[7] As before, we might allow a generic subscript z such that any $/n/[s]$ is $/n/[s]_z$ and $\sim/n/[s]$ is $\neg\overline{/n/[s]}_z$. Then the negation rules apply as stated.

$$\begin{array}{l|l} \textbf{R} & /P/_s \\ & \\ & /P/_s \end{array}$$

$$\begin{array}{l|l} \neg\textbf{I} & \begin{array}{|l} /P/_s \\ \hline \\ /\!/Q/\!/_t \\ \backslash\!\backslash\neg Q\backslash\!\backslash_t \end{array} \\ & \backslash\neg P\backslash_s \end{array}$$

$$\begin{array}{l|l} \neg\textbf{E} & \begin{array}{|l} /\neg P/_s \\ \hline \\ /\!/Q/\!/_t \\ \backslash\!\backslash\neg Q\backslash\!\backslash_t \end{array} \\ & \backslash P\backslash_s \end{array}$$

$$\begin{array}{l|l} \wedge\textbf{I} & /P/_s \\ & /Q/_s \\ & \\ & /P \wedge Q/_s \end{array}$$

$$\begin{array}{l|l} \wedge\textbf{E} & /P \wedge Q/_s \\ & \\ & /P/_s \end{array}$$

$$\begin{array}{l|l} \wedge\textbf{E} & /P \wedge Q/_s \\ & \\ & /Q/_s \end{array}$$

$$\begin{array}{l|l} \vee\textbf{I} & /P/_s \\ & \\ & /P \vee Q/_s \end{array}$$

$$\begin{array}{l|l} \vee\textbf{I} & /P/_s \\ & \\ & /Q \vee P/_s \end{array}$$

$$\begin{array}{l|l} \vee\textbf{E} & /P \vee Q/_s \\ & \begin{array}{|l} /P/_s \\ \hline \\ /\!/R/\!/_t \end{array} \\ & \begin{array}{|l} /Q/_s \\ \hline \\ /\!/R/\!/_t \end{array} \\ & /\!/R/\!/_t \end{array}$$

$$\begin{array}{l|l} \supset\textbf{I} & \begin{array}{|l} /P/_s \\ \hline \\ \backslash Q\backslash_s \end{array} \\ & \backslash P \supset Q\backslash_s \end{array}$$

$$\begin{array}{l|l} \supset\textbf{E} & \backslash P \supset Q\backslash_s \\ & /P/_s \\ & \\ & \backslash Q\backslash_s \end{array}$$

$$\begin{array}{l|l|l} \rightarrow\textbf{E} & /s.t.u/ & /s.t.u/ \\ & /P \rightarrow Q/_s & /P \rightarrow Q/_s \\ & P_t & \overline{P}_u \\ & & \\ & Q_u & \overline{Q}_t \end{array}$$

$$\begin{array}{l|l|l} \rightarrow\textbf{I} & \begin{array}{|l} /s.t.u/ \\ P_t \\ \hline \\ Q_u \end{array} & \begin{array}{|l} /s.t.u/ \\ \overline{P}_u \\ \hline \\ \overline{Q}_t \end{array} \\ & /P \rightarrow Q/_s & /P \rightarrow Q/_s \end{array}$$

where t and u do not appear in any undischarged premise or assumption

$$\begin{array}{l|l|l} \textbf{CL} & /n/[s] & /n/[s] \\ & /\!/P/\!/_s & \\ & & \\ & \backslash\!\backslash P\backslash\!\backslash_s & \backslash n\backslash[s] \end{array}$$

$$\begin{array}{l|l} \textbf{NI} & \\ & \\ & \\ & n[0] \end{array}$$

$$\begin{array}{l|l} \textbf{NE} & /n/[s] \\ & s.t \\ & \\ & /n/[t] \end{array}$$

$$\begin{array}{l|l} \textbf{0I} & /n/[a] \\ & \\ & \\ & /a.s.s/ \end{array}$$

$$\begin{array}{l|l|l} \textbf{0E} & /n/[a] & /n/[a] \\ & /a.s.t/ & /a.s.t/ \\ & \mathcal{P}(s) & \mathcal{P}(t) \\ & & \\ & \mathcal{P}(t) & \mathcal{P}(s) \end{array}$$

These are the rules for the base systems. DW takes all the rules but CL. Roy's FA systems drop the NI, NE, 0I and 0E rules. FB systems are like DW except that they may add CL. It is then possible to add from the following in the natural way.

$\Box$I
$$\begin{array}{l} \quad s.t \\ \quad \vdash \\ \quad /P/_t \\ /\Box P/_s \end{array}$$
where t does not appear in any undischarged premise or assumption

$\Box$E
$$\begin{array}{l} /\Box P/_s \\ s.t \\ /P/_t \end{array}$$

AMρ
$$\begin{array}{l} s.s \end{array}$$

AMσ
$$\begin{array}{l} s.t \\ t.s \end{array}$$

AMτ
$$\begin{array}{l} s.t \\ t.u \\ s.u \end{array}$$

AM11
$$\begin{array}{l} /s.t.u/ \\ \quad /s.t.y/ \\ \quad y.t.u \\ \quad \vdash \\ \quad /\!/P/\!/_w \\ /\!/P/\!/_w \end{array} \qquad \begin{array}{l} /s.t.u/ \\ \quad /s.y.u/ \\ \quad \overline{y.t.u} \\ \quad \vdash \\ \quad /\!/P/\!/_w \\ /\!/P/\!/_w \end{array}$$

$\leq$E
$$\begin{array}{l} a \leq b \\ P_a \\ P_b \end{array}$$

$\leq^*$E
$$\begin{array}{l} a \leq^* b \\ P_a \\ \overline{P}_b \end{array}$$

$\leq^\sharp$E
$$\begin{array}{l} a \leq^\sharp b \\ \overline{P}_a \\ P_b \end{array}$$

AM$^9_{10}$
$$\begin{array}{l} /a.b.x/ \\ x.c.d \\ \quad b.c.y \\ \quad /a.y.d/ \\ \quad \vdash \\ \quad /\!/P/\!/_w \\ /\!/P/\!/_w \end{array} \qquad \begin{array}{l} /a.b.x/ \\ x.c.d \\ \quad b.y.d \\ \quad /a.c.y/ \\ \quad \vdash \\ \quad /\!/P/\!/_w \\ /\!/P/\!/_w \end{array} \qquad \begin{array}{l} /a.x.b/ \\ \overline{x.c.d} \\ \quad \overline{b.c.y} \\ \quad /a.y.d/ \\ \quad \vdash \\ \quad /\!/P/\!/_w \\ /\!/P/\!/_w \end{array} \qquad \begin{array}{l} /a.x.b/ \\ \overline{x.c.d} \\ \quad \overline{b.y.d} \\ \quad /a.c.y/ \\ \quad \vdash \\ \quad /\!/P/\!/_w \\ /\!/P/\!/_w \end{array}$$

AM12
$$\begin{array}{l} a.b.c \\ \quad y \geq a \\ \quad b.y.c \\ \quad \vdash \\ \quad /\!/P/\!/_w \\ /\!/P/\!/_w \end{array} \qquad \begin{array}{l} a.b.c \\ \quad y \geq^* a \\ \quad \overline{c.b.y} \\ \quad \vdash \\ \quad /\!/P/\!/_w \\ /\!/P/\!/_w \end{array} \qquad \begin{array}{l} \overline{a.b.c} \\ \quad y \geq^\sharp a \\ \quad b.y.c \\ \quad \vdash \\ \quad /\!/P/\!/_w \\ /\!/P/\!/_w \end{array} \qquad \begin{array}{l} \overline{a.b.c} \\ \quad y \leq a \\ \quad \overline{c.b.y} \\ \quad \vdash \\ \quad /\!/P/\!/_w \\ /\!/P/\!/_w \end{array}$$
where y does not appear in any undischarged premise or assumption and is not w

Though they will not play a natural role in most derivations, we also allow the following rules for inclusion relations:

$\leq$E: $a \leq b,\ b.x.y,\ n[a] \vdash x \leq y$; $a \leq b,\ b.x.y,\ \sim n[a] \vdash a.x.y$
$a \leq b,\ \overline{a.x.y},\ \overline{n}[b] \vdash x \leq y$; $a \leq b,\ \overline{a.x.y},\ \sim\overline{n}[b] \vdash \overline{b.x.y}$

$\leq^*$E: $a \leq^* b,\ \overline{b.x.y},\ n[a] \vdash x \leq y$; $a \leq^* b,\ \overline{b.x.y},\ \sim n[a] \vdash a.x.y$
$a \leq^* b,\ \overline{a.x.y},\ n[b] \vdash x \leq y$; $a \leq^* b,\ \overline{a.x.y},\ \sim n[b] \vdash b.x.y$

$\leq^\sharp$E: $a \leq^\sharp b,\ b.x.y,\ \overline{n}[a] \vdash x \leq y$; $a \leq^\sharp b,\ b.x.y,\ \sim\overline{n}[a] \vdash \overline{a.x.y}$
$a \leq^\sharp b,\ a.x.y,\ \overline{n}[b] \vdash x \leq y$; $a \leq^\sharp b,\ a.x.y,\ \sim\overline{n}[b] \vdash \overline{b.x.y}$

Where Γ is a set of unsubscripted formulas, let Γ_0 be those same formulas, each with subscript 0. Then,

VNR4 $\Gamma \vdash_{NR4x} A$ iff there is an *NR4x* derivation of A_0 from the members of Γ_0.

Examples. Here are some cases, with the first ones paired to illustrate the match between derivations that do, and ones that do not, include the NI, NE, 0I and 0E rules.

A5 $\vdash_{NR4x} [(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

1	0.1.2	A (g, →I)
2	$(A \to B) \wedge (A \to C)_1$	
3	$n[0]$	NI
4	2.3.4	A (g, →I)
5	A_3	
6	$(A \to B) \wedge (A \to C)_2$	3,1,2 0E
7	$(A \to B)_2$	6 ∧E
8	B_4	4,7,5 →E
9	$(A \to C)_2$	6 ∧E
10	C_4	4,9,5 →E
11	$(B \wedge C)_4$	∧I
12	$[A \to (B \wedge C)]_2$	4-11 →I
13	$[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]_0$	1-12 →I

$(A \to B) \wedge (A \to C) \vdash_{NR4Ax} A \to (B \wedge C)$

1	$(A \to B) \wedge (A \to C)_0$	P
2	0.1.2	A (g, →I)
3	A_1	
4	$(A \to B)_0$	1 ∧E
5	$(A \to C)_0$	1 ∧E
6	B_2	2,4,3 →E
7	C_2	2,5,3 →E
8	$(B \wedge C)_2$	6,7 ∧I
9	$A \to (B \wedge C)_0$	2-8 →I

A8 $\vdash_{NR4x} (A \to \neg B) \to (B \to \neg A)$

1	0.1.2	A (g, →I)
2	$(A \to \neg B)_1$	
3	$n[0]$	NI
4	2.3.4	A (g, →I)
5	B_3	
6	$\overline{A}_4$	A (c, ¬I)
7	$(A \to \neg B)_2$	3,1,2 0E
8	$\overline{\neg B}_3$	4,7,6 →E
9	B_3	5 R
10	$\neg A_4$	6-9 ¬I
11	$(B \to \neg A)_2$	4-10 →I
12	$(A \to \neg B) \to (B \to \neg A)_0$	1-11 →I

$A \to \neg B \vdash_{NR4Ax} B \to \neg A$

1	$(A \to \neg B)_0$	P
2	0.1.2	A (g, →I)
3	B_1	
4	$\overline{A}_2$	A (c, ¬I)
5	$\overline{\neg B}_1$	2,1,4 →E
6	B_1	3 R
7	$\neg A_2$	4-6 ¬I
8	$(B \to \neg A)_0$	2-7 →I

$\vdash_{NR4r} (\neg A \to A) \to A$

1	0.1.2	A (g, $\to$I)
2	$(\neg A \to A)_1$	
3	$n[0]$	NI
4	0.3.2	A (g, 1 AM11)
5	$\overline{3.1.2}$	
6	$4 \geq^\sharp 3$	A (g, 5 AM12)
7	1.4.2	
8	$\overline{\neg A}_2$	A (c, $\neg$E)
9	$\overline{\neg A}_3$	3,4,8 0E
10	$\neg A_4$	6,9 $\leq^\sharp$E
11	A_2	7,2,10 $\to$E
12	$\overline{\neg A}_2$	8 R
13	A_2	8-12 $\neg$E
14	A_2	5,6-13 AM12
15	A_2	1,4-14 AM 11
16	$[(\neg A \to A) \to A]_0$	1-15 $\to$I

$A \to B \vdash_{NR4m} \Box A \to \Box B$

1	$(A \to B)_0$	P
2	0.1.2	A (g, $\to$I)
3	$\Box P_1$	
4	$n[0]$	NI
5	2.3	A (g, $\Box$I)
6	$\Box P_2$	4,2,3 0E
7	P_3	5,6 $\Box$E
8	0.3.3	4 0I
9	Q_3	8,1,7 $\to$E
10	$\Box Q_2$	5-9 $\Box$I
11	$(\Box P \to \Box Q)_0$	2-10 $\to$I

10 Many-Valued Modal Logics: $\boldsymbol{K}_{Lx}$ (appendix)

This section is developed again in in terms as for section 7. This smooths presentation, and applies to Priest as before.

10.1 Language / Semantic Notions

LK_L The VOCABULARY consists of propositional parameters $p_0, p_1 \ldots$ with the operators $\neg$, $\wedge$, $\vee$, $\Box$, and $\Diamond$. Each propositional parameter is a

FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $\Box A$, and $\Diamond A$. $A \supset B$ abbreviates $\neg A \vee B$. If A is a formula so formed, so is $\overline{A}$.

Let $/A/$ and $\backslash A\backslash$ represent either A or $\overline{A}$ where what is represented is constant in a given context, but $/A/$ and $\backslash A\backslash$ are opposite.

IK_L An INTERPRETATION is $\langle W, R, h\rangle$ where W is a set of worlds; $R \subseteq R^2$ is a modal access relation; and $h_w(/p/) = 0$ or $h_w(/p/) = 1$. Optionally interpretations are subject to,

exc for no p are both $h_w(p) = 1$ and $h_w(\overline{p}) = 0$

exh for any p either $h_w(p) = 1$ or $h_w(\overline{p}) = 0$

ρ Reflexivity: for all x, xMx.

σ Symmetry: for all x, y if xMy then yMx.

τ Transitivity: for all x, y, z if xMy and yMz then xMz.

We get K_{LP} with *exh*, and K_{K3} with *exc*. K_{FDE} has neither of these constraints. We recover classical K with both. These logics may add ρ, σ and τ in the natural way.

HK_L For complex expressions,

($\neg$) $h_w(/\neg P/) = 1$ iff $h_w(\backslash P\backslash) = 0$

($\wedge$) $h_w(/P \wedge Q/) = 1$ iff $h_w(/P/) = 1$ and $h_w(/Q/) = 1$

($\vee$) $h_w(/P \vee Q/) = 1$ iff $h_w(/P/) = 1$ or $h_w(/Q/) = 1$

($\Box$) $h_w(/\Box P/) = 1$ iff there is no $x \in W$ such that wMx and $h_x(/P/) = 0$

($\Diamond$) $h_w(/\Diamond P/) = 1$ iff there is some $x \in W$ such that wMx and $h_x(/P/) = 1$

For a set Γ of formulas, $h_w(\Gamma) = 1$ iff $h_w(/P/) = 1$ for each $/P/ \in \Gamma$; then,

VK_L $\Gamma \models_{K_{Lx}} P$ iff there is no K_{Lx} interpretation $\langle W, R, h\rangle$ and w such that $h_w(\Gamma) = 1$ but $h_w(P) = 0$.

10.2 Natural Derivations: $\boldsymbol{NK}_L$

Derivations combine methods from modal and multi-valued logics in the natural way. Allow subscripts to indicate worlds. (D) corresponds to *exc* and (U) to *exh*.

$$\textbf{D}\ \begin{array}{|l} P_s \\ \\ \overline{P}_s \end{array} \qquad \textbf{R}\ \begin{array}{|l} /P/_s \\ \\ /P/_s \end{array} \qquad \neg\textbf{I}\ \begin{array}{|l} \begin{array}{|l} /P/_s \\ \hline \\ /\!/Q/\!/_t \\ \backslash\!\backslash\neg Q\backslash\!\backslash_t \end{array} \\ \backslash\neg P\backslash_s \end{array} \qquad \neg\textbf{E}\ \begin{array}{|l} \begin{array}{|l} /\neg P/_s \\ \hline \\ /\!/Q/\!/_t \\ \backslash\!\backslash\neg Q\backslash\!\backslash_t \end{array} \\ \backslash P\backslash_s \end{array}$$

$$\textbf{U}\ \begin{array}{|l} \overline{P}_s \\ \\ P_s \end{array} \qquad \wedge\textbf{I}\ \begin{array}{|l} /P/_s \\ /Q/_s \\ \\ /P \wedge Q/_s \end{array} \qquad \wedge\textbf{E}\ \begin{array}{|l} /P \wedge Q/_s \\ \\ /P/_s \end{array} \qquad \wedge\textbf{E}\ \begin{array}{|l} /P \wedge Q/_s \\ \\ /Q/_s \end{array}$$

$$\vee\textbf{I}\ \begin{array}{|l} /P/_s \\ \\ /P \vee Q/_s \end{array} \qquad \vee\textbf{I}\ \begin{array}{|l} /P/_s \\ \\ /Q \vee P/_s \end{array} \qquad \vee\textbf{E}\ \begin{array}{|l} /P \vee Q/_s \\ \begin{array}{|l} /P/_s \\ \hline \\ /\!/R/\!/_t \end{array} \\ \begin{array}{|l} /Q/_s \\ \hline \\ /\!/R/\!/_t \end{array} \\ /\!/R/\!/_t \end{array}$$

$$\supset\textbf{I}\ \begin{array}{|l} \begin{array}{|l} /P/_s \\ \hline \\ \backslash Q\backslash_s \end{array} \\ \backslash P \supset Q\backslash_s \end{array} \qquad \supset\textbf{E}\ \begin{array}{|l} \backslash P \supset Q\backslash_s \\ /P/_s \\ \\ \backslash Q\backslash_s \end{array}$$

$$\textbf{AM}\rho\ \begin{array}{|l} \\ \\ s.s \end{array} \qquad \textbf{AM}\sigma\ \begin{array}{|l} s.t \\ \\ t.s \end{array} \qquad \textbf{AM}\tau\ \begin{array}{|l} s.t \\ t.u \\ \\ s.u \end{array}$$

$$\Box\textbf{I}\ \begin{array}{|l} \begin{array}{|l} s.t \\ \hline \\ /P/_t \end{array} \\ /\Box P/_s \end{array} \qquad \Box\textbf{E}\ \begin{array}{|l} /\Box P/_s \\ s.t \\ \\ /P/_t \end{array} \qquad \Diamond\textbf{I}\ \begin{array}{|l} /P/_t \\ s.t \\ \\ /\Diamond P/_s \end{array} \qquad \Diamond\textbf{E}\ \begin{array}{|l} /\Diamond P/_s \\ \begin{array}{|l} s.t \\ /P/_t \\ \hline \\ /\!/Q/\!/_u \end{array} \\ /\!/Q/\!/_u \end{array}$$

($\Box$I) where t does not appear in any undischarged premise or assumption

($\Diamond$E) where t does not appear in any undischarged premise or assumption and is not u

Every subscript is 0, appears in a premise, or in the t place of an assumption for $\Box$I or $\Diamond$E. Where the members of Γ and A are without overlines or subscripts, let Γ_0 be the members of Γ, each with subscript 0. Then, NK$_{\text{L}}$ $\Gamma \vdash_{NK_{Lx}} A$ iff there is an NK_{Lx} derivation of A_0 from Γ_0.

We allow standard two-way derived rules (including MN) with overlines and subscripts constant throughout. MT, NB and DS appear in the forms,

MT	**NB**		**DS**	
$/P \supset Q/_s$	$/P \equiv Q/_s$	$/P \equiv Q/_s$	$/P \vee Q/_s$	$/P \vee Q/_s$
$\backslash\neg Q\backslash_s$	$\backslash\neg P\backslash_s$	$\backslash\neg Q\backslash_s$	$\backslash\neg P\backslash_s$	$\backslash\neg Q\backslash_s$
$/\neg P/_s$	$/\neg Q/_s$	$/\neg P/_s$	$/Q/_s$	$/P/_s$

Examples. The first couple cases are matched to show an equivalent result by different means.

$\Box A \wedge \neg\Box B \vdash_{NK_{FDE}} \Diamond(A \wedge \neg B)$

1	$(\Box A \wedge \neg B)_0$	P
2	$\Box A_0$	1 $\wedge$E
3	$\neg\Box B_0$	1 $\wedge$E
4	$\Diamond\neg B_0$	3 MN
5	$\quad$ 0.1	A (g 4$\Diamond$E)
6	$\quad \neg B_1$	
7	$\quad A_1$	2,5 $\Box$E
8	$\quad (A \wedge \neg B)_1$	7,6 $\wedge$I
9	$\quad \Diamond(A \wedge \neg B)_0$	5,8 $\Diamond$I
10	$\Diamond(A \wedge \neg B)_0$	4,5-9 $\Diamond$E

$\Box A \wedge \neg\Box B \vdash_{NK_{FDE}} \neg\Box\neg(A \wedge \neg B)$

1	$(\Box A \wedge \neg\Box B)_0$	P
2	$\Box A_0$	1 $\wedge$E
3	$\quad \overline{\Box\neg(A \wedge \neg B)}_0$	A (c, $\neg$I)
4	$\quad\quad$ 0.1	A (g, $\Box$I)
5	$\quad\quad\quad \neg B_1$	A (c, $\neg$E)
6	$\quad\quad\quad A_1$	2,4 $\Box$E
7	$\quad\quad\quad (A \wedge \neg B)_1$	6,5 $\wedge$I
8	$\quad\quad\quad \overline{\neg(A \wedge \neg B)}$	3 $\Box$E
9	$\quad\quad \overline{B}_1$	5-8 $\neg$E
10	$\quad \overline{\Box B}_0$	5-9 $\Box$I
11	$\quad \neg\Box B_0$	1 $\wedge$E
12	$\neg\Box\neg(A \wedge \neg B)_0$	3-11 $\neg$I

$\Box A \vdash_{NK_{LP\tau}} \Box\Box A$

1	$\overline{\Box A}_0$	A (g, $\supset$I)
2	0.1	A (g, $\Box$I)
3	1.2	A (g, $\Box$I)
4	0.2	2,3 AMτ
5	$\overline{A}_2$	1,4 $\Box$E
6	A_2	5 U
7	$\Box A_1$	3-6 $\Box$I
8	$\Box\Box A_0$	2-7 $\Box$I
9	$(\Box A \supset \Box\Box A)_0$	1-8 $\supset$I

$\Box(\Diamond A \supset B) \vdash_{NK_{FDE\sigma\tau}} \Box(A \supset \Box B)$

1	$\Box(\Diamond A \supset B)_0$	P
2	0.1	A (g, $\Box$I)
3	$\overline{A}_1$	A (g, $\supset$I)
4	1.2	A (g, $\Box$I)
5	2.1	4 AMσ
6	$\overline{\Diamond A}_2$	3,5 $\Diamond$I
7	0.2	2,5 AMτ
8	$(\Diamond A \supset B)_2$	1,7 $\Box$E
9	B_2	8,6 $\supset$E
10	$\Box B_1$	4-9 $\Box$I
11	$(A \supset \Box B)_1$	3-10 $\supset$I
12	$\Box(A \supset \Box B)_0$	2-11 $\Box$I

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